GRAPH THEORY

STUDY MATERIAL

B.Sc. MATHEMATICS

VI SEMESTER

ELECTIVE COURSE

(2011 ADMISSION)

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

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Contents

Module 1

Graphs and Subgraphs

1.1 Introduction

Graph theory is a branch of mathematics which deals the problems, with the help of diagrams. There are may applications of graph theory to a wide variety of subjects which include operations research, physics, chemistry, computer science and other branches of science. In this chapter we introduce some basic concepts of graph theory and provide variety of examples. We also obtain some elementary results.

1.2 What is a graph ?

Definition 1.2.1. A *graph* G consists of a pair $(V(G), X(G))$ where $V(G)$ is a non empty finite set whose elements are called **points or vertices** and $X(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $X(G)$ are called **lines or edges** of the graph G. If $x = \{u, v\} \in X(G)$, the line x is said to join u and v. We write $x = uv$ and we say that the points u and v are **adjacent**. We also say that the point u and the line x are incident with each other. If two lines x and y are incident with a common point then they are called **adjacent lines**. A graph with p points and q lines is called a (p, q) *graph*. When there is no possibility of confusion we write $V(G) = V$ and $X(G) = X$.

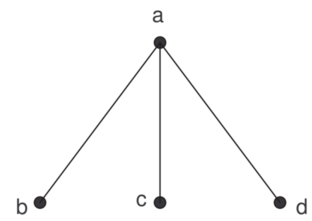

Figure 1.1: A an example of a $(4, 3)$ graph

1.3 Representation of a graph

It is customary to represent a graph by a diagram and refer to the diagram itself as the graph. Each point is represented by a small dot and each line is represented by a line segment joining the two points with which the line is incident. Thus a diagram of graph depicts the incidence relation holding between its points and lines. In drawing a graph it is immaterial whether the lines are drawn straight or curved, long or short and what is important is the incidence relation between its points and lines.

Example 1.3.1.

1. Let $V = \{a, b, c, d\}$ and $X = \{\{a, b\}, \{a, c\}\{a, d\}\}$, $G = (V, X)$ is a $(4, 3)$ graph. This graph can be represented by the diagram given in figure 1.1. In this graph the points a and b are adjacent whereas b and c are nonadjacent.

2. Let $V = \{1, 2, 3, 4\}$ and $X = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. Then $G = (V, X)$ is a $(4, 6)$ graph. This graph is represented by the diagram given in figure 1.2 Although the lines $\{1, 2\}$ and $\{2, 4\}$ intersect in the diagram, their intersection is not a point of the graph. Figure 1.3 is another diagram for the graph given in figure 1.2.

3. The $(10, 15)$ graph given in figure 1.4 is called the **Petersen graph.**

Remark 1.3.1. The definition of a graph does not allow more than one line joining two points. It also does not allow any line joining a point to itself. Such a line joining a point to itself is called a **loop.**

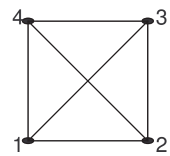

Figure 1.2: An example of a $(4, 6)$ graph

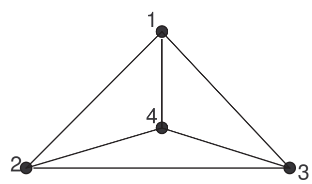

Figure 1.3: Another representation of graph shown in figure 1.1

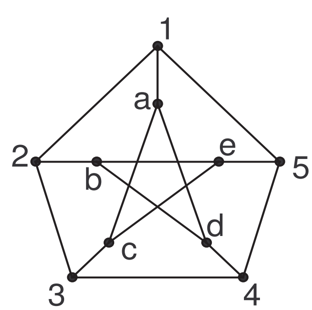

Figure 1.4: Peterson graph

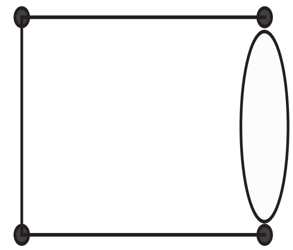

Figure 1.5: A multiple graph

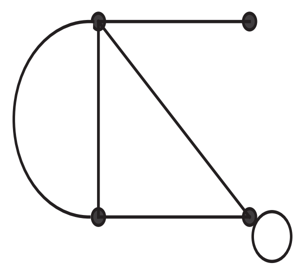

Figure 1.6: A pseudograph

Definition 1.3.1. If more than one line joining two vertices are allowed, the resulting object is called a **multigraph**. Line joining the same points are called **multi lines**. If further loops are also allowed, the resulting object is called **Pseudo graph**.

Example 1.3.2. Figure1.5 is a multigraph and figure 1.6 is a pseudo graph.

Remark 1.3.2. Let G be a (p, q) graph. Then $q \leqslant \begin{pmatrix} p \\ 2 \end{pmatrix}$ and $q = \begin{pmatrix} p \\ 2 \end{pmatrix}$ iff any two distinct points are adjacent.

Definition 1.3.2. A Graph in which any two distinct points are adjacent is called a **complete graph**. The complete graph with p points is denoted by K_p. K_3 is called a triangle. The graph given Fig. 1.3 is K_4 and K_5 is shown in Fig.1.7

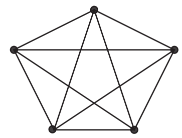

Figure 1.7: K_5

Definition 1.3.3. A graph whose edge set is empty is called a **null graph** or a **totally disconnected graph**.

Definition 1.3.4. A graph G is called labeled if its p points are distinguished from one another by names such as $v_1, v_2 \cdots v_p$.
The graphs given in Fig. 1.1 and Fig. 1.3 are labelled graphs and the graph in Fig. 1.7 is an unlabelled graph.

Definition 1.3.5. A graph G is called a **bigraph** or **bipartite graph** if V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joins a point of V_1 to a point of V_2. $(V_1, V_2)$is called a **bipartition** of G. If further G contains every line joining the points of V_1 to the points of V_2 then G is called a **complete bigraph**. If V_1 contains m points and V_2 contains n points then the complete bigraph G is denoted by $K_{m,n}$. The graph given in Fig. 1.1 is $K_{1,3}$. The graph given in Fig. 1.8 is $K_{3,3}$. $K_{1,m}$ is called a **star** for $m \geq 1$.

1.4 Exercise

1. Draw all graphs with 1, 2, 3 and 4 points.

2. Find the number of points and lines in $K_{m,n}$.

3. Let $V = \{1, 2, 3, \cdots, n\}$. Let $X = \Big\{ \{i, j\} | \ i, j \in V$and are relatievly prime$\Big\}$. The resulting graph (V, X) is denoted by G_n. Draw G_4 and G_5.

1.5 Degrees

Definition 1.5.1. The **degree** of a point v_i in a graph G is the number of lines incident with v_i .The degree of v_i is denoted by $d_G(v_i)$ or deg v_i or $d(v_i)$.

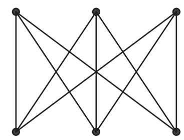

Figure 1.8: bigraph

A point v of degree 0 is called an **isolated point**. A point v of degree 1 is called an endpoint.

Theorem 1.5.1. The sum of the degrees of the points of a graph G is twice the number of lines. That is,$\sum_i$ deg$v_i = 2q$.

Proof. Every line of G is incident with two points. Hence every line contribute 2 to the sum of the degrees of the points. Hence $\sum_i degv_i = 2q$. □

Corollary 1.5.1. In any graph G the number of points of odd degree is even.

Proof. Let $v_1, v_2, \cdots, v_k$ denote the point of odd degree and $w_1, w_2 \cdots, w_m$ denote the points of even degree in G. By theorem 1.5.1, $\sum_{i=1}^{k} \deg(v_i) + \sum_{i=1}^{w} \deg w_i = 2q$ which is even. Further $\sum_{i=1}^{m} \deg w_i$ is even. Hence $\sum_{i=1}^{m} \deg v_i$ is also even. But degv_i is odd for each i. Hence k must be even. □

Definition 1.5.2. For any graph G,we define

$$\delta(G) = \min\{\deg v / v \in V(G)\} \text{ and}$$
$$\Delta(G) = \max\{\deg v / v \in V(G)\}.$$

It all the points of G have the same degree r, then $\delta(G) = \Delta(G) = r$ and this case G is called a **regular graph** of degree r. A regular graph of degree 3 is called a cubic graph. For example, the complete graph K_p is regular of degree $p - 1$.

Theorem 1.5.2. Every cubic graph has an even number of points.

Proof. Let G be a cubic graph with p points, then $\sum \deg v = 3p$ which is even by theorem 1.5.1. Hence p is even. □

1.6 Solved Problems

Problem 1. Let G be a (p, q) graph all of whose points have degree k or $k+1$. If G has $t > 0$ points of degree k, show that $t = p(k+1) - 2q$.

Solution

Since G has t points of degree k, the remaining $p - t$ points have degree $k+1$. Hence $\sum_{v \in V} d(v) = tk + (p-t)(k+1)$.

$$\therefore\ tk + (p-t)(k+1) = 2q$$
$$\therefore t = p(k+1) - 2q.$$

Problem 2. Show that in any group of two or more people, there are always two with exactly the same number of friends inside the group.

Solution. We construct a graph G by taking the group of people as the set of points and joining two of them if they are friends, then deg$v$ is equal to number of friends of v and hence we need only to prove that at least two points of G have the same degree. Let $V(G) = \{v_1, v_2, \cdots, v_p.\}$. Clearly $0 \leq \deg v_i \leq p-1$ for each i. Suppose no two points of G have the same degree. Then the degrees of $v_1, v_2, \cdots, v_p$. are the integers $0, 1, 2, \cdots, p-1$ in some order. However a point of degree $p-1$ is joined to every other point of G and hence no point can have degree zero which is a contradiction. Hence there exist two points of G with equal degree.

Problem 3. Prove that $\delta\ \leq 2q/p \leq\ \Delta$

Solution

Let $V(G) = \{v_1, v_2, \cdots, v_p\}$. We have $\delta\ \leq\ \deg v_i \leq\ \Delta$. for all i. Hence

$$p\delta \leq \sum_{i=1}^{p} \deg v_i \leq p\Delta.$$

$$\therefore\ p\delta\ \leq\ 2q\ \leq\ p\Delta \text{ (by theorem2.1)}$$
$$\therefore \delta\ \leq\ \frac{2q}{p} \leq \Delta$$

Problem 4. Let G be a k-regular bibgraph with bipartion (V_1, V_2) and $k > 0$. Prove that $|V_1| = |V_2|$.

Solution

Since every line of G has one end in V_1 and other end in V_2 it follows that

$\sum_{v\in V_1} d(v) = \sum_{v\in V_2} d(v) = q$. Also $d(v) = k$ for all $v \in V = V_1 \cup V_2$. Hence $\sum_{v\in V_1} d(v) = k|V_1|$ and $\sum_{v\in V_2} d(v) = k|V_2|$ so that $k\ |V_1| = k|V_2|$. Since $k > 0$, we have $|V_1| = |V_2|$.

1.7 Exercise

1. Given an example of a regular graph of degree 0
2. Give three examples for a regular graph of degree 1
3. Give three examples for a regular graph of degree 2
4. What is the maximum degree of any point in a graph with p points?
5. Show that a graph with p points is regular of degree $p - 1$ if and only if it is complete
6. Let G be a graph with at least two points show that G contains two vertices of the same degree
7. A (p, q) graph has t points of degree m and all other points are of degree n. Show that $(m - n)t + pn = 2q$.

1.8 Subgraphs

Definition 1.8.1. A graph $H = (V_1, X_1)$ is called **subgraph** of $G = (V, X)$. $V_1 \subseteq V$ and $X_1 \subseteq X$. If H is a subgraph of G we say that G is a **supergraph** of H. H is called a **spanning subgraph** of G if H is the maximal subgraph of G with point set V_1. Thus, if H is an induced subgraph of G, two points are adjacent in H they are adjacent in G. If $V_2 \subseteq V$, then the induced subgraph of G induced by V_2 and is denoted by $G[X]$. If $X_2 \subseteq X$, then the sub graph of G with line set X_2 and is denoted by $G[X_2]$

Examples. Consider the petersen graph G given in Fig. 1.4. The graph given in Fig.1.9 is a subgraph of G. The graph given in Fig.1.10 is an induced subgraph of G. The graph given in Fig.1.10 is an induced subgraph of G. The graph given in Fig1.11 is a spanning subgraph of G.

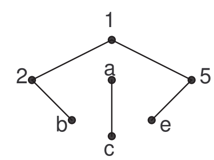

Figure 1.9: Subgraph

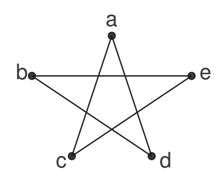

Figure 1.10: Induced subgraph

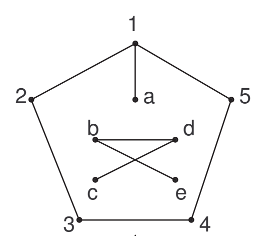

Figure 1.11: Spanning subgraph

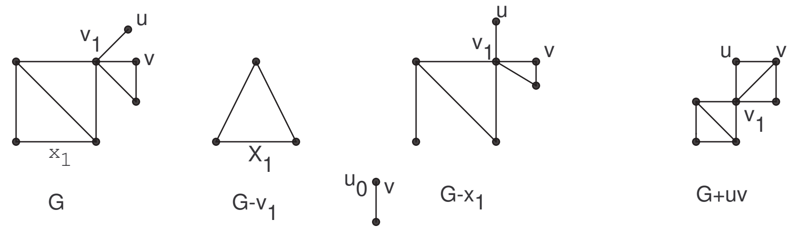

Figure 1.12:

Definition 1.8.2. Let $G = (V, X)$ be a graph.Let $v_i \in V$. The subgraph of G obtained by removing the point v_i and all the lines incident with v_i is called the **subgraph obtained by the removal of the point** v_i and is denoted by G-v_i. Thus if $G - v_i = (V_i, X_i)$ then $V_i = V - v_i$ and $X_i = \{x/x \in X$ and x is not incident with $v_i\}$. Clearly $G - v_i$ is an induced subgraph of G. Let $x_i \in X$. Then $G - x_i = (V, X - x_j)$ is called the subgraph of G obtained by the removal of the line x_j. Clearly $G - x_j$ is a spanning subgraph of G which contains all the lines of G except x_j. The removal of a set of points or lines from G is defined to be the removal of single elements in succession.

Definition 1.8.3. Let $G = (V, X)$ be a graph. Let v_i, v_j be two points which are not adjacent in G. Then $G + v_iv_j = (V, X \bigcup \{v_i, v_j\})$ is called the graph obtained by **the addition of the line** v_iv_j to G

Clearly $G+v_iv_j$ is the smallest super graph of G containing the line v_iv_j.We listed these concepts in Fig1.12. The proof given in the following theorem is typical of several proofs in theory.

Theorem 1.8.1. The maximum number of lines among all p point graph no triangles is $\left[\frac{p^2}{4}\right]$. ([$x$] denotes the greatest integer not exceeding the the real number x).

Proof. The result can be easily verified for $p \leq 4$. For $p > 4$, we will prove by induction separately for odd p and for every p.

Part 1. For odd p.

Suppose the result is true for all odd $p \leq 2n + 1$. Now let G be a (p, q) graph with $p = 2n + 3$ and no triangles. If$q = 0$, then $q \leq \left[\frac{p^2}{4}\right]$. Hence let $q > 0$. Let u and v be a pair of adjacent points. The subgraph $G' = G - \{u, v\}$ has

$2n+1$ points and no triangles. Hence induction hypothesis,

$$q(G') \leq \left[\frac{(2n+1)^2}{4}\right] = \left[\frac{4n^2+4n+1}{4}\right]$$
$$= \left[n^2+n+\frac{1}{4}\right] = n^2+n$$

Since G has no triangles, no point of G' can be adjacent to both u and G. Now, lines in G are of three types.

1. Lines of $G'(\leq n^2+n\, in\, number\, by (1))$

2. Lines between G' and $\{u,v\}(\leq 2n+1\, innumberby(2))$

3. Line uv

Hence

$$\begin{aligned} q \leq (n^2+n)+(2n+1)+1 &= n^2+3n+2 \\ &= \frac{1}{4}(4n^2+12n+8) \\ &= \left(\frac{4n^2+12n+9}{4}-\frac{1}{4}\right) \\ &= \left[\frac{(2n+3)^2}{4}\right] = \left[\frac{p^2}{4}\right] \end{aligned}$$

Also for $p=2n+3$, the graph $K_{n+1,n+2}$ has no triangles and has $(n+1)(n+2) = n^2+3n+2 = [\frac{p^2}{4}]$ lines. Hence this maximum q is attained.

Part 2. For even p.

Suppose the result is true for all even $p \leq 2n$. Now let G be a (p,q) graph with $p=2n+2$ and no triangles. As before, let u and v be a pair of adjacent points in G and let $G' = G-\{u,v\}$.

Now G' has $2n$ points and no triangles. Hence by hypothesis,

$$q(G') \leq \left[\frac{(2n)^2}{4}\right] = n^2$$

Lines in G are of three types.

(i) Lines of G'

(ii) Lines between G' and $\{u, v\}$

(iii) line uv.

Hence $q \leq n^2 + 2n + 1 = (n+1)^2 = \frac{(2n+2)^2}{4} = [p^2/4]$. Hence the result holds for even p also. We see that for $p = 2n+2$. $K_{n+1,n+1}$ is a $(p, [\frac{p^2}{4}]$ graph without triangles. □

1.9 Exercise

1. Show that $K_p - v = K_{p-1}$ for any point v of K_p.
2. Show that an induced subgraph of a complete graph is complete.
3. Let $G = (V, X)$ be a (p, q) graph. Let $v \in V$ and $x \in X$. Find the number of points and lines in $G - v$ and $G - x$.
4. If every induced proper subgraph of a graph G is complete and $p > 2$ then show that G is complete.
5. If every induced proper subgraph of a graph G is totally disconnected, then show that G is totally disconnected.
6. Show that in a graph G every induced graph is complete iff every induced graph with two points is complete.

1.10 Isomorphism

Definition 1.10.1. Two graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ are said to be **isomorphic** if there exists a bijection $f : V_1 \to V_2$ such that u, v are adjacent in G_1 if and only if $f(u), f(v)$ are adjacent in G_2. If G_1 is isomorphic to G_2, we write $G_1 \cong G_2$. The map f is called an isomorphism from G_1 to G_2.

Example 1.10.1. 1. The graph given in Fig. 2.2 and Fig. 2.3 are isomorphic.

2. The two graphs given in Fig.1.13 are isomorphic. $f(u_i) = v_i$ is an isomorphism between these two graphs.

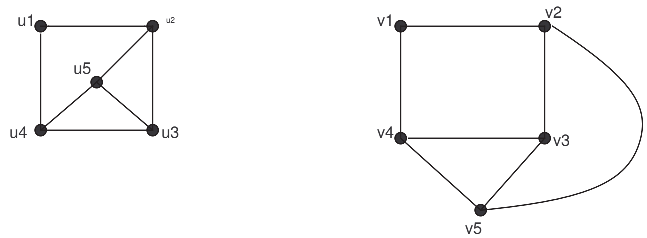

Figure 1.13:

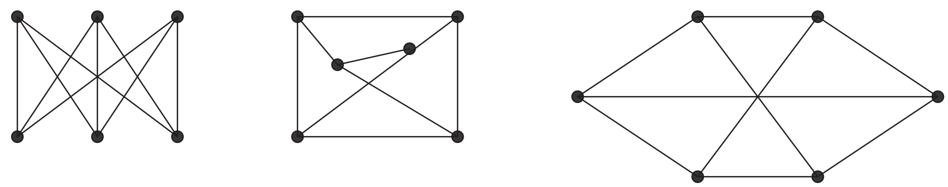

Figure 1.14:

3. The three graphs given in Fig.1.14 are isomorphic with each other.

Theorem 1.10.1. Let f be an isomorphism of the graph $G_1 = (V_1, X_1)$ to the graph $G_2 = (V_2, X_2)$. Let $v \in V_1$. Then deg $v =$ deg $f(v)$. i.e., isomorphism preserves the degree of vertices.

Proof. A point $u \in V_1$ is adjacent to v in G_1 iff $f(u)$ is adjacent to $f(v)$ in G_2. Also f is bijection. Hence the number of points in V_1 which are adjacent to v is equal to the number of points in V_2 which are adjacent to $f(v)$. Hence deg $v =$ deg $f(v)$. □

Remark 1.10.1. Two isomorphic graphs have the same number of points and the same number of lines. Also it follows from Theorem 1.10.1that two isomorphic graphs have equal number of points with a given degree. However these conditions are not sufficient to ensure that two graphs are isomorphic. For example consider the two graphs given in figure 1.15. By theorem 1.10.1, under any isomorphism w_4 must correspond to v_3; w_1, w_5, w_6 must correspond to v_1, v_5, v_6 in some order. The remaining two points w_2, w_3 are adjacent whereas v_2, v_4 are not adjacent. Hence there does not exist an isomorphism

Figure 1.15:

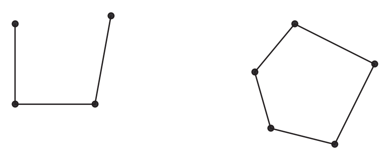

Figure 1.16:

between these two graphs. However both graphs have exactly one vertex of degree 3, three vertices of degree 1 and two vertices of degree 2.

Definition 1.10.2. An isomorphism of a graph G onto itself is called an **automorphism** of G.

Remark 1.10.2. Let $\Gamma(G)$ denote the set of all automorphism of G. Clearly the identity map $i : V \to V$ defined by $i(v) = v$ is an automorphism of G so that $i \in \Gamma(G)$. Further if α and β are automorphisms of G then $\alpha.\beta$ and α^{-1} are also automorphism of G. Hence $\Gamma(G)$ is a group and is called the **automorphism group** of G.

Definition 1.10.3. Let $G = (V, X)$ be a graph. The **complement** $\overline{G}$ of G is defined to be the graph which has V as its set of points and two points are adjacent in $\overline{G}$ iff they are not adjacent in G. G is said to be a **self complementary** graph if G is isomorphic to $\overline{G}$.

For example the graphs given in Fig.1.16 are self complementary graphs.

It has been conjectured by Ulam that the collection of vertex deleted subgraphs $G - v$ determines G upto isomorphism.

Solved Problems

Problem 5. Prove that any self complementary graphs has $4n$ or $4n+1$ points

Solution. Let $G = (V(G), X(G))$ be a self complementary graph with p points.
Since G is self complementary, G is isomorphic to $\overline{G}$.
$\therefore |X(G)| = |X(\overline{G})|$. Also

$$\begin{aligned} |X(G)| + |X(\overline{G})| &= \begin{pmatrix} p \\ 2 \end{pmatrix} = \tfrac{p(p-1)}{2} \\ \therefore\ 2|X(G)| &= \frac{p(p-1)}{2} \\ \therefore |X(G)| &= \frac{p(p-1)}{4} \text{ is an integer.} \end{aligned}$$

Further one of p or $p-1$ is odd. Hence p or $p-1$ is a multiple of 4. $\therefore p$ is of the the form $4n$ or $4n+1$.

Problem 6. Prove that $\Gamma(G) = \Gamma(\overline{G})$.

Solution. Let $f \in \Gamma(G)$ and let $u, v \in V(G)$.

$$\begin{aligned} \text{Then}\, u, v\, \text{are adjacent } in\ \overline{G} &\Leftrightarrow u, v \text{ are not adjacent in } G. \\ &\Leftrightarrow f(u), f(v) \text{ are not adjacent in } G \\ \text{(since } f \text{ is an automorphism of } G) & \\ &\Leftrightarrow f(u), f(v) \text{ are adjacent in } \overline{G}. \end{aligned}$$

Hence f is an automorphism of $\overline{G}$.

$\therefore f \in \Gamma(\overline{G})$ and hence $\Gamma(G) \subseteq \Gamma(\overline{G})$.

Similarly $\Gamma(\overline{G}) \subseteq \Gamma(G)$ so that $\Gamma(G) = \Gamma(\overline{G})$.

1.11 Exercise

1. Prove that any graph with p points is isomorphic to a subgraph of K_p.
2. Show that isomorphism is an equivalence relation among graphs.
3. Show that the two graphs given in Fig. 2.17 are not isomorphic.

4. Show that upto isomorphism there are exactly four graphs on three vertices.

5. Prove that a graph G is complete iff $\overline{G}$ is totally disconnected.

6. Let G be (p, q) graph $deg_{\overline{G}}(v) = p - 1 - deg_G(v)$.

7. Prove that $\Gamma(K_n) \cong S_n$, the symmetric group of degree n.

1.12 Ramsey Numbers

We start by considering the following puzzle. In any set of six people there will always be either a subset of three who are mutually acquainted, or a subset of three who are mutually strangers. This situation may be represented by a graph G with six points representing the six people in which adjacency indicates acquaintances. The above puzzle then asserts that G contains three mutually adjacent points or three mutually non-adjacent points. Equivalently G or $\overline{G}$ contains a triangle.

Theorem 1.12.1. For any graph G with 6 points, G or $\overline{G}$ contains a triangle.

Proof. Let v be a point of G. Since G contains 5 points other than v, v must be either adjacent to three points in G or non-adjacent to three points in G.Hence v must be adjacent to three points either in G or in $\overline{G}$ Without loss of generality, let us assume that v is adjacent to three points u_1, u_2, u_3 in G. If two of these three points are adjacent, G contains a triangle. Otherwise these three points from a triangle in $\overline{G}$. Hence G or $\overline{G}$ contains a triangle. □

It is easy to see that the above theorem is not true for graphs with less than 6 points and we have this as an exercise to the reader. Thus 6 is the smallest positive integer such that any graph G on 6 points contains K_3 or $\overline{K_3}$. This suggests the following general question. What is the least positive integer $r(m, n)$ such that for any graph G with $r(m, n)$ points, G contains K_m or $\overline{K_n}$. For example $r(3, 3) = 6$. The numbers $r(m, n)$ are called Ramsey numbers after F. Ramsey who proved the existence of $r(m, n)$. The determination of the Ramsey numbers is difficult unsolved problem. **Solved Problems**

Problem 7. Prove that $r(m, n) = r(n, m)$.

Solution Let $r(m, n) = s$. Let$G$ be any graph on s points. Then $\overline{G}$ also has s points. Since $r(m, n) = s$,$\overline{G}$ has either K_m or $\overline{K_n}$ as an induced subgraph. Hence G has K_n or $\overline{K_m}$ as an induced subgraph. Thus an arbitrary graph on s points contains K_n or $\overline{K_m}$ as an induced subgraph. $\therefore r(n, m) \leq s$. i.e, $r(n, m) \leq r(m, n)$. Interchanging m and n we get $r(m, n) \leq r(n, m)$. Hence $r(m, n) = r(n, m)$.

Problem 8. Prove that $r(2, 2) = 2$

Solution Let G be a graph on 2 points. Let $V(G) = \{u, v\}$. Then u and v are either adjacent in G or adjacent in $\overline{G}$. Hence G or $\overline{G}$ contains K_2. Thus if G is any graph on two points, then G or $\overline{G}$ contains K_2 and clearly 2 is the least positive integer with this property. Hence $r(2, 2) = 2$.

1.13 Exercise

1. Prove, by suitable examples, that theorem 1.12.1 is not true graphs with less than 6 points.
2. Find $r(1, 1)$.
3. Find $r(k, 1)$ for any positive integer k.
4. Find $r(2, 3)$.
5. Find $r(2, k)$ for any positive integer k.

1.14 Indepedent Sets and Coverings

Definition 1.14.1. A covering of a graph $G = (V, X)$ is a subset K of V such that every line of G is incident with a vertex in K. A covering K is called a **minimum covering** if G has no covering K' with $|K'| < |K|$. The number of vertices in a minimum covering of G is called the **covering number** of G and is denoted by β.

A subset S of V is called an **independent set** of G if no two vertices S are adjacent in G. An independent set S is said to be **maximum** if G has no independent set S' with $|S'| > |S|$. The number of vertices in a maximum independent set is called **independence number** of G and is denoted α.

Example

Consider the graph given in Fig. 1.18 $\{v_6\}$ is an independent set. $\{v_1, v_3\}$ is a maximum independent set. $\{v_1, v_2, v_3, v_4, v_5\}$ is a covering and $\{v_2, v_3, v_4, v_5\}$ is a minimum covering.

Theorem 1.14.1. A set $S \subseteq V$ is an independent set of G if and only if V is a covering of G.

Proof. By definition, S is independent iff no two vertices of S are adjacent.That is, iff every line of S is incident with at least one point of $V - S$. That is, iff $V - S$. is a covering of G. □

Corollary 1.14.1. $\alpha + \beta = p$

Proof. Let S be a maxium independent set of G and K be a minimum covering of G.
$\therefore |S| = \alpha$ and $|K| = \beta$.
Now $V - S$ is a covering of G and K is a minimum covering of G. Hence $|K| \leq |V - S|$ so that $\beta \leq p - \alpha$

$$\therefore \beta + \alpha \leq p \tag{1.1}$$

Also $V - K$ is an independent set and S is a maximum independent set
Hence $|S| \leq |V - K|$ so that $\alpha \geq p - \beta$.

$$\alpha + \beta \geq p \tag{1.2}$$

From 1.1 and (1.2) , we get $\alpha + \beta = p$.

In the following definition we give the line analogue of coverings independence. □

Definition 1.14.2. A line covering of G is a subset L of X such that every vertex is incident with a line of L. The number of line in a minimum line covering of G is called the **line covering number** of G and is denoted

by β'. A set of lines is called **independent** if no two of them are adjacent. The number of lines in a maximum independent set of lines is called the **edge independence number** and is denoted by α'. Gallai has proved that for any non-trivial graph, $\alpha' + \beta' = p$, though it is not true that the complement of an independent set of lines is a line covering.
Result $\alpha' + \beta' = p$.

Proof. Let S be a maximum independent set of lines of G so that $|S| = \alpha'$. Let M be a set of lines, one incident for each of the $p - 2\alpha'$ points of G not covered by any line of S. Clearly $S \bigcup M$ is a line covering of G.

$$\therefore |S \cup M| \geq \beta'$$
$$\therefore \alpha' + P - 2\alpha' \geq \beta'$$
$$\therefore p \geq \alpha' + \beta' \tag{1.3}$$

Now, let T be a minimum line cover of G, so that $|T| = \beta'$. T cannot have a line x both of whose ends are also incident with lines of T other than x (since, otherwise $T - \{x\}$ will become a line covering of G). Hence $G|T|$, the spanning subgraph of G induced by T, is the union of stars. Hence each line of T is incident with at least one endpoint of $G[T]$. Let W be a set of endpoints of $G[T]$ consisting of exactly one end point for each line of T. Hence $|W| = |T| = \beta'$ and each star has exactly one point not in W . Hence

$$p = |W| + (number\ of\ stars\ in\ G[T]) \tag{1.4}$$
$$\therefore p = \beta' + (number\ of\ stars\ in\ G[T]) \tag{1.5}$$

By choosing one line from each star of $G[T]$, we get set of independent lines of G. Hence

$$\alpha' \geq (number\ of\ stars\ in\ G[T])$$

Hence (1.5) gives $p \leq \beta' + \alpha'$.
Therefore by ((1.3)), $\alpha' + \beta' = p$. This complete the proof. □

1.15 Exercise

1. Find α, β, α' and β' for the complete graph K_p.
2. Prove or disprove. Every covering of a graph contains a minimum cover.
3. Prove or disprove. Every independent set of lines is contained in a maximum independent set of lines.
4. Give an example to show that the complement of an independent set of lines need not be a line covering.
5. Give an example to show that the complement of a line covering need be an independent set of lines.

1.16 Intersection graphs and line graphs

Definition 1.16.1. Let $F = \{S_1, S_2, \cdots, S_p\}$be a non- empty family of distinct non empty subsets of a given set S. The **intersection graph** of F, denoted $\Omega(F)$ is defined as follows:
The set of points V of $\Omega(F)$ is F itself and two points S_i, S_j are adjacent if $i \neq j$ and $S_i \bigcap S_j \neq \emptyset$. A graph G is called an intersection graph on S if there exist a family F of subsets of S such that G is isomorphic to $\Omega(F)$.

Theorem 1.16.1. Every graph is an intersection graph.

Proof. Let $G = (V, X)$ be a graph. Let $V = \{v_1, v_2, \cdots, v_p\}$. Let $S = V \cup X$ For each $v_i \in V$, let $S_i = \{v_i\} \cup \{x \in X | v_i \in x\}$.

Cleary $F = \{S_1, S_2, \cdots, S_p\}$ is a family of distinct non-empty subsets of S

Further if v_i, v_j are adjacent in V then $v_i v_j \in Si \cap S_j$ and hence $Si \cap S_j \neq \emptyset$. Conversly if $Si \cap S_j \neq \emptyset$ then the element common to $Si \cap S_j$ is the line joining v_i and v_j so that v_i, v_j are adjacent in G. Thus $f : V \to F$ defined by $f(v_i) = S_i$ is an isomorphism of G to $\Omega(F)$. Hence G is an intersection graph. □

Definition 1.16.2. Let $G = (V, X)$ be a graph with $X \neq \emptyset$. Then X can be thought of a family of 2 element subsets of V. The intersection graph $\Omega(X)$ is called the **line graph** of G and is denoted by $L(G)$. Thus the points of $L(G)$ are lines of G and two points in $L(G)$ are adjacent iff the corresponding lines are adjacent in G.

A example of a graph and line graph are given in Fig.1.19.

Theorem 1.16.2. Let G be a (p, q) graph. $L(G)$ is a (q, q_L) graph where $q_L = \frac{1}{2}(\sum_{i=1}^{p} {d_i}^2) - q$.

Proof. By definition, number of points in $L(G)$ is q. To find the number of lines in $L(G)$. Any two of the d_i lines incident with v_i are adjacent in $L(G)$ and hence we get $\frac{d_i(d_i-1)}{2}$ lines in $L(G)$.

$$\begin{aligned} Hence\ q_L &= \sum_{i=1}^{p} \frac{d_i(d_i-1)}{2} \\ &= \frac{1}{2}(\sum_{i=1}^{p} {d_i}^2) - \frac{1}{2}(\sum_{i=1}^{p} d_i) \\ &= \frac{1}{2}(\sum_{i=1}^{p} {d_i}^2) - \frac{1}{2}(2q) \\ &= \frac{1}{2}(\sum_{i=1}^{p} {d_i}^2) - q \end{aligned}$$

□

1.17 Exercise

Show that the line graphs of the two graphs given in Fig.1.20 are isomorphic.

The two graphs given in figure 2.20 constitute the only pair of non-isomorphic connected graphs having isomorphic line graphs. In all other cases, $L(G) \cong L(G')$ implies $G \cong G'$ as claimed in the following theorem.

Theorem 1.17.1. **(Whitney.)** Let G and G' be connected graphs with isomorphic line graphs. Then G and G' are isomorphic unless one is K_3 and the other $K_{1,3}$.

Definition 1.17.1. A Graph G is called a **line graph** if $G \cong L(H)$ for some graph H.

Example $K_4 - x$ is a line graph as seen in Fig.1.19. The following theorem is called Beineke's forbidden subgraph characteristics of line graphs.

Theorem 1.17.2. (Beineke.) G is a line graph iff none of the nine graphs of Fig. 2.20 is an induced subgraph of G.

1.18 Operations on graphs

Definition 1.18.1. Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two graphs with $V_1 \cap V_2 = \Phi$. We define:

- The **union** $G_1 \cup G_2$ to be (V, X) where

$$V = V_1 \cup V_2 \textit{ and } X = X_1 \cup X_2$$

  .

- The **sum** $G_1 + G_2$ as $G_{1 \cup G_2}$ together with all the lines joining points of V_1 to points of V_2.

- The **product** $G_1 \times G_2$ having $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent to v_1 in G_1 and $u_2 = v_2$.

- The **composition** $G_1[G_2]$ as having $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if u_1 is adjacent to v_1 in G_1 or ($u_1 = v_1$ and u_2 is adjacent to v_2 in G_2).

We note that $\overline{K_m} + \overline{K_n} = K_{m,n}$.

Theorem 1.18.1. Let G_1 be a (p_1, q_1) and G_2 a (p_2, q_2) graph.

1. $G_1 \cup G_2$ is a $(p_1 + p_2, q_1 + q_2)$ graph.
2. $G_1 + G_2$ is a $(p_1 + p_2, q_1 + q_2 + p_1 p_2)$ graph.
3. $G_1 \times G_2$is a $(p_1 p_2, q_1 p_2 + q_2 p_1)$ graph.
4. $G_1[G_2]$ is $(p_1 p_2, p_1 q_2 + p_2^2 q_1)$ graph.

Proof.

1. is obvious.

2.

$$\begin{aligned}
number\ of\ lines\ in\ G_1 + G_2 &= number\ of\ lines\ in\ G_1 + number\ of\ lines\ in G_2 \\
&+ number\ of\ lines\ joining\ points\ of\ V_1\ of\ points\ of V_2. \\
&= q_1 + q_2 + p_1 p_2.\ Hence\ we\ get\ (2)
\end{aligned}$$

3. Clearly number of points in $G_1 \times G_2$ is $p_1 p_2$.
   Now, let $(u_1, u_2) \in V_1 \times V_2$. The points adjacent to (u_1, u_2) are (u_1, v_2) where u_2 is adjacent to v_2 (v_1, u_2) where adjacent to u_1.

$$\therefore deg(u_1, u_2) = deg u_1 + deg u_2$$

   The total number of lines in $G_1 \times G_2$

$$\begin{aligned}
&= \frac{1}{2}[\sum_{i,j} \mathrm{deg}(u_i) + deg(v_j)] \\
&= \frac{1}{2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}(\mathrm{deg} u_i + deg v_j)\ where\ u_i \in V_1, v_j \in V_2 \\
&= \frac{1}{2}\sum_{i=1}^{p_1}(p_2 \mathrm{deg} u_i + \sum_{j=1}^{p_2} \mathrm{deg} v_j) \\
&= \frac{1}{2}\sum_{i=1}^{p_1}(p_2 \mathrm{deg} u_i + 2q_2) \\
&= \frac{1}{2}(2p_2 q_1 + 2p_1 q_2) \\
&= p_2 q_1 + p_1 q_2
\end{aligned}$$

The proof of (4) is left to the reader. □

1.19 Exercise

1. Prove (4) of Theorem1.17.1.

2. If G_1 and G_2 are regular, determine whether $G_1 + G_2, G_1 \times G_2$ and G_1 are regular.

3. What is $K_m + K_n$?

4. Express $K_4 - x$ in terms of K_2 and $\overline{K_2}$.

5. Express the graph in Fig. 2.21 in terms of $\overline{K_3}$ and $\overline{K_2}$.

6. Express the graph G of Fig. 2.19 in terms of K_1 and K_3.

7. Define two more binary operations on graphs in your own way.

Revision Questions Determine which of the following statements are true and which are false.

1. If G is a (p, q) graph $q \leq \begin{pmatrix} p \\ 2 \end{pmatrix}$

2. If G is a (p, q) graph and $q = \begin{pmatrix} p \\ 2 \end{pmatrix}$ then G is complete.

3. A subgraph of a complete graph is complete.

4. An induced subgraph of a complete graph is complete.

5. A subgraph of a bipartite graph is bipartite.

6. In any graph G the number of points of odd degree is even.

7. Any complete graph is regular.

8. Any complete bigraph is regular.

9. A regular graph of degree 0 is totally disconnected.

10. The only regular graph of degree 1 is K_2.

11. The only connected regular graph of degree i is K_2.

12. A graph G is regular iff $\delta = \Delta$.

13. An induced subgraph of regular graph is regular.

14. If G is regular, then $G - V$ is regular.

15. If G is complete, then $G - V$ is complete.

16. Any two isomorphic graphs have the same number of points and same number of lines.

17. Any two graphs having the same number of points and same number of lines are isomorphic.

18. Isomorphism preserves the degree of vertices.

19. If G_1 and G_2 are regular, $G_1 + G_2$ is regular.

20. If G_1 and G_2 are regular $G_1[G_2]$ is regular.

Answers
$1, 2, 4, 5, 6, 7, 9, 11, 12, 15, 16, 18$ and 20 are true.

1.20 Walks, Trails and Paths

Definition 1.20.1. A **walk** of a graph G is an alternating sequence of points and lines $v_0, x_1, v_1, x_2, v_2, \cdots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each line x_i is incident with v_{i-1} and v_i.

We say that the walks join v_0 and v_n and it is called a v_0-v_n walk. v_0 is called the **initial point** and v_1 is called the **terminal point** of the walk. The above walk is also denoted by $v_0, v_1, \cdots, v_n$ the lines of the walks being self evident. n, the number of lines in the walk, is called the length of this walk. A single point is considered as a walk of length 0. A walk is called a **trail** if all its lines are distinct and is called a **path** if all its points are distinct.

Example 1.20.1. For the graph given in 1.23 $v_1, v_2, v_3, v_4, v_2, v_1, v_2, v_5$ is a walk. $v_1, v_2, v_4, v_3, v_2, v_5$ is a trail but not a path. v_1, v_2, v_4, v_5 is a path. Obliviously, every path is a trail and a trail need not be a path.
The graph consisting of a path with n points is denoted by P_n.

Definition 1.20.2. A $v_0 - v_n$ walk is called **closed** if $v_0 = v_n$. A closed walk $v_0, v_1, \cdots, v_n = v_0$ in which $n \geqslant 3$ and $v_0, v_1, \cdots, v_{n-1}$ are is distinct is called of length n. A graph consisting of a **cycle** of length n is denoted by C_n.
C_3 is called a **triangle**.

Theorem 1.20.1. In a graph G, any $u - v$ walk contains a $u - v$ path.

Proof. We prove the result by induction on the length of the walk. Any walk of length 0 or 1 is obviously a path. Now, assume the result for all walks of length less than n. If $u = u_0, u_1, \cdots, u_n = v$ be a $u - v$ walk of length n. If all the points of the walk are distinct it is already a path. If not, there exists i and j such that $0 \leq i < j \leq n$ and $u_i = u_j$. Now $u = u_0, \cdots, u_i, u_{j+1}, \cdots, u_n = v$ is a $u - v$ walk of length less than n which by induction hypothesis contains a $u - v$ path. □

Theorem 1.20.2. If $\delta \geq k$, then G has a path of length k.

Proof. Let v_1 be an arbitrary point.Choose v_2 adjacent to v_1. Since $\delta \geq k$, there exists at least $k - 1$ vertices other than v_1 which are adjacent to v_2. Choose $v_1 \neq v_1$ such that v_3 is adjacent to v_2. In general having chosen $v_1, v_2, \cdots, v_i$ where $1 < i \leq \delta$ there exist a point $v_{i+1} \neq v_0, v_1, \cdots, v_n$ such that v_{i+1} is adjacent to v_i. This process yields a path of length k in G.
Aliter. Let $P = (v_0, v_1, \cdots, v_n)$ be the longest path in G. Then every vertex adjacent to v_0 lies on P. Since$d(v_0) \geq \delta$ it follows that length of $P \geq \delta \geq k$. Hence $P_1 = (v_0, v_1, \cdots, v_k)$ is a path of length k in G. □

Theorem 1.20.3. A closed walk of odd length contains a cycle.

Proof. Let $v = v_0, v_1, \cdots, v_n = v$ be a closed walk of odd length. Hence $n \geq 3$. If $n = 3$ this walk is itself the cycle C_3 and hence the result is trivial. Now assume the result for all walks of length less than n. If the given walk of length n is itself is a cycle there is nothing to prove.If not there exists two positive integers i and j such that $i < j, \{i, j\} \neq \{0, n\}$ and $v_i = v_j$. Now $v_i, v_{i+1}, \cdots, v_j$ and $v = v_0, v_1, \cdots, v_i, v_{j+1}, \cdots, v_n = v$ are closed walks contained in the given walk and the sum of their lengths is n. Sin ce n is odd at least one of these walks is of odd length which by induction hypothesis contains a cycle. □

Solved Problem

Problem 9. If A is the adjacency matrix of a graph with $V = \{v_1, v_2, \cdots, v_p\}$,prove that for any $n \geq 1$ the $(i, j)^{th}$ entry of A^n is the number of $v_i - v_j$ walks of length n in G.

Solution We prove the result by induction on n. The number of $v_i - v_j$ walks of length 1

$$= \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$
$$= a_{ij}.$$

Hence the result is true for $n = 1$.

We now assume that the result is true for $n - 1$. Let $A^{n-1} = (a_{ij}^{(n-1)})$ so that $a_{ij}^{(n-1)}$ is number of $v_i - v_j$ walks of length $n - 1$ in G. Now $A^{n-1}A = (a_{ij}^{(n-1)})a_{ij}$.Hence $(i, j)^th$ entry of

$$A^n = \sum_{k=1}^{p} a_{ik}^{(n-1)} a_{kj} \tag{1.6}$$

Also every $v_i - v_j$ walk of length n in$G$ consists of a $v_i - v_j$ walk of length $n - 1$ followed by a vertex v_j which is adjacent to v_k. Hence v_j is adjacent to v_k then $a_{kj} = 1$ and $a_{ij}^{(n-1)}$ represents the number of $v_i - v_j$ walks of length n whose last edge is $v_i v_j$. Hence the right hand side of equation (1.6) gives the number of $v_i - v_j$ walks of length n in G. This completes the induction and the proof.

1.21 Connectness and components

Definition 1.21.1. Two points u and v of a graph G are said to be **connected** if there exists a $u - v$ path in G.

Definition 1.21.2. A graph G is said to be **connected** if every pair of its points are connected. A graph which is not connected is said to be **disconnected.**
For example, for $n > 1$ the graph $\overline{K_n}$ consisting of n points and no lines is disconnected. The union of two graphs is disconnected.

It is an easy exercise to verify that connectedness of points is an equivalence relation on the set of points V. Hence v is partitioned into nonempty subsets $V_1, V_2, \cdots, V_n$ such that two vertices $u\ and\ v$ are connected iff both u and v belongs to the same set V_i.Let G_i denote the induced subgraph of G with vertex set V_i. Clearly the subgraphs $G_1, G_2, \cdots, G_n$ are connected and are called the **Components of G**.

Clearly a graph G is connected iff it has exactly one component. 1.24 gives a disconnected graph with 5 components.

Theorem 1.21.1. A graph G with p points and $\delta \geq \frac{p-1}{2}$ is connected.

Proof. Suppose G is not connected. Then G has more than one component. Consider any component $G_1 = (V_1, X_1)$ of G. Let $v_1 \in V_1$. Since $\delta \geq \frac{p-1}{2}$ there exist at least $\frac{p-1}{2}$ points in G_1 adjacent to v_1 and hence V_1 contains at least $\frac{p-1}{2} + 1 = \frac{p+1}{2}$ points. Thus each component of G contains at least $\frac{p+1}{2}$ points and G has at least two components. Hence number of points in $G \geq p + 1$ which is a contradiction. Hence G is connected. □

Theorem 1.21.2. A graph G is connected iff for any partition of V into subsets V_1 and V_2 there is a line of G joining a point of V_1 to a point of V_2.

Proof. Suppose G is connected.Let $V = V_1 \cup V_2$ be a partition of a V into two subset. Let $u \in V_1$ and $v \in V_2$. Since G is connected, there exists a $u - v$ path in G, say, $u = v_0, v_1, v_2, \cdots, v_n = v$. Let i be the least positive integer such that $v_i \in V_2$.(Such an i exists since $v_n = v \in V_2$). Then $v_{i-1} \in V_1$ and v_{i-1}, v_i are adjacent. Thus there is a line joining $v_{i-1} \in V_1$ and $v_i \in V_2$. To prove the converse, suppose G is not connected. Then G contains at least two components. Let V_1 denote the set of all vertices of one component and V_2 the remaining vertices of G. Clearly $V = V_1 \cup V_2$ is a partition of V and there is no line joining any point of V_1 to any point of V_2. Hence the theorem. □

Theorem 1.21.3. If G is not connected then $\overline{G}$ is connected.

Proof. Since G is not connected, G has more than one component.Let u, v be any two points of G. We will prove that there is a $u-v$ path in $\overline{G}$. If u, v belong to different components in G, they are not adjacent in G and hence they are adjacent in $\overline{G}$.If u, v lie in the same component of G, choose w in a different component. Then u, w, v is a $u - v$ path in $\overline{G}$. Hence $\overline{G}$ is connected. □

Definition 1.21.3. For any two points u, v of a graph we define the distance between u and v by $d(u,v) = \begin{cases} \text{the length of the shortest } u - v \text{ path ,} & \text{if such a path exists;} \\ \infty, & \text{otherwise.} \end{cases}$
If G is a connected Graph, $d(u, v)$ is always a non-negative integer. In this case d is actually a metric on the set of points V(See problem 2).

Theorem 1.21.4. A graph G with at least two points is bipartite iff all its cycles are of even length.

Proof. Suppose G is a bipartite. Then V can be partitioned into two subsets V_1 and V_2 such that every line joins a point of V_1 to a point of V_2. Now consider any cycle $v_0, v_1, v_2, \cdots, v_n = v_0$ of length n. Suppose $v_0 \in V_1$. Then $v_2, v_4, v_6 \cdots \in V_1$ and $v_1, v_3, v_5 \cdots \in V_2$.Further $v_n = v_0 \in V_1$ and hence n is even. Conversely,suppose all cycles in G are of even length. We may assume without loss of generality that G is connected.(If not we consider the components of G separately). Let $v_1 \in V$. Define

$$\begin{aligned} V_1 &= \{v \in V | d(v, v_1) \text{ is even}\} \\ V_2 &= \{v \in V | d(v, v_1) \text{ is odd}\}. \end{aligned}$$

Clearly, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$. We claim that every line of G joins a point of V_1 to a point of V_2. Suppose two points $u, v \in V_1$ are adjacent. Let p be a shortest $v_1 - u$ path of length m and let Q be a shortest $v_1 - v$ path of length n. Since $u, v \in V_1$ both m and n are even. Now, let u_1 be the last point common to P and Q. Then the $v_1 - u_1$ path along P and the $v_1 - u_1$ path along Q are both shortest path and hence have the same length, say i. Now the $u_1 - u$ path along P, the line uv followed by the $v - u_1$ path along Q form a cycle of length $(m - i) + 1 + (n - i) = m + n - 2i + 1$ which is odd and this is a contradiction. Thus no two points of V_1 are adjacent. Similarly no two points of V_2 are adjacent and hence G is bipartite. Hence the theorem. □

To study the measure of connectedness of a graph G we consider the minimum number of points or lines to be removed from the graph in order to disconnect it.

Definition 1.21.4. A **cut point** of a graph G is a point whose removal increases the number of components.A **bridge** of a graph G is a line whose removal increases the number of components.

Clearly if v is a cut point of a connected graph, $G - v$ is disconnected. For the graph given in Fig.1.25,1,2, and 3 are cut points. The lines $\{1, 2\}$ and $\{3, 4\}$ are bridges. 5 is non-cut point.

Theorem 1.21.5. Let v be a point of a connected graph G. The following statements are equivalent.

1. v is a cut-point of G.
2. There exists a partition of $V - \{v\}$ into subsets U and W such that for each $u \in U$ and $w \in W$, the point v is on every $u - w$ path.
3. There exists two points u and w distinct from v such that v is on every $u - w$ path.

Proof. (1) $\Rightarrow$ (2). Since v is a cut-point of G, $G - v$ is disconnected. Hence $G - v$ has at least two components. Let U consist of the points of one of the components of $G - v$ and W consist of the points of the remaining components. Clearly $V - \{v\} = U \cup W$ is a partition of $V - \{v\}$. Let $u \in U$ and $w \in W$. Then u and w lie in different components of $G - v$. Hence there is no $u - w$ path in $G - v$.
Therefore every $u - w$ path in G contains in v.
(2) $\Rightarrow$ (3). This is trivial.
(3) $\Rightarrow$ (1). Since v is on every $u - w$ path in G there is no $u - w$ path in $G - v$. Hence $G - v$ is not connected so that v is a cut point of G.

□

Theorem 1.21.6. Let x be a line of a connected graph G. The following statements are equivalent.

1. x is bridge of G.
2. There exists a partition of V into two subsets U and W such that for every point $u \in U$ and $w \in W$, the line x is on every $u - w$ path.
3. There exists two points u, w such that the line x is on every $u - w$ path.

Proof. The proof is analogous to that of theorem 1.21.5 and is left as an exercise. □

Theorem 1.21.7. A line x of a connected graph G is a bridge iff x is not on any cycle of G.

Proof. Let x be a bridge of G. Suppose x lies on a cycle C of G. Let w_1 and w_2 be any two points in G. Since G is connected, there exists a $w_1 - w_2$ path P in G. If x is not on P, then P is a path in $G - x$. If x is on P, replacing x by $C - x$, we obtain a $w_1 - w_2$ walk in $G - x$. Walk contains a $w_1 - w_2$ path in $G - x$.Hence $G - x$ is connected which is contradiction to (1). Hence x is not on any cycle on G. Conversely, let $x = uv$ be not on any cycle of G. Suppose x is not a bridge. Hence $G - x$ is connected.
$\therefore$ There is a $u - v$ path in $G - x$. This path together with the line $x = uv$ forms a cycle containing x and contradicts (2).Hence x is a bridge. □

Theorem 1.21.8. Every non-trivial connected graphs has at least two points which are not cut points.

Proof. Choose two points u and v such that $d(u, v)$ is maximum. We claim that u and v are not cut points. Suppose v is a cut point.Hence $G - v$ has more than one component. Choose a point w in a component that does not contain u.Then v lies on every $u - w$ path and hence $d(u, w) > d(u, v)$ which is impossible.Hence v is not a cut point. Similarly u is not a cut point. Hence the theorem. □

1.22 Exercise

1. Prove that connectedness of points is an equivalence relation on the points of G.

2. Prove that for a connected graph G the distance function $d(u, v)$ is actually a metric on G. i.e, $d(u, v) \geq 0$ and $d(u, v) = 0$ iff $u = v, d(u, v) = d(v, u)$ and $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in V$.

3. Prove theorem 4.9.

4. If $x = uv$ is a bridge for a connected graph $G \neq K_2$, show that either u or v is a cut point of G.

5. Prove that if x is a bridge of a connected graph G, then $G - x$ has exactly two components. Give an example to show that a similar result is not true for a cut point.

6. The girth of a graph is defined to be the length of its shortest cycle. Find the girths of (i) K_m $(ii)K_{m,n}$ $(iii)C_n$ (iv) The Peterson graph.

7. The circumference of a graph is defined to be the length of its longest cycle. Find the *circumference* of the graphs given in problem 6.

8. Prove that if G is connected then its line graph is also connected.

9. Prove that any graph G with $\delta \geq r \geq 2$ contains a cycle of length at least $r + 1$.

10. Prove that if there exists two distinct cycles each containing a line x, then there exists a cycle not containing x.

11. Prove that if a graph G has exactly two points of odd degree there must be a path joining these two points.

12. Give an example of a connected graph in which every line is a bridge.

13. Prove that any graph with p points satisfying the conditions of problem 12 must have exactly $p - 1$ lines.

14. Give an example of a graph which has a cut point but does not have a bridge.

15. Prove that if v is a cut point of G, then v is not a cut point of $\overline{G}$

1.23 Blocks

Definition 1.23.1. A connected non-trivial graph having no cut point is a **block**. A block of a graph is a subgraph that is a block and is maximal with respect to this property.

A graph and its blocks are given in 1.26. In the following theorem we give several equivalent conditions for a given block.

Theorem 1.23.1. Let G be a connected graph with at least three point,following statements are equivalent.

1. G is a block.

2. Any two points of G lie on a common cycle.

3. Any point and any line of G lie on a common cycle.

4. Any two lines of G lie on a common cycle.

Proof. (1) $\Rightarrow$ (2) Suppose G is a block. We shall prove by induction on the distance $d(u, v)$ between u and v any two vertices u and v lie on a common cycle. Suppose $d(u, v) = 1$. Hence u and v are adjacent. By hypothesis, $G \neq K_2$ and G has no cut points. Hence the line $x = uv$ is not a bridge and Theorem 1.21.7 x is on a cycle of G. Hence the points u and v lie on a common cycle of G. Now assume that the result is true for any two vertices at distance k and let $d(u, v) = k \geq 2$. Consider a $u - v$ path of length k. Let w be the vertex that precedes v on this path. Then $d(u, v) = k - 1$. Hence by induction hypothesis there exists a cycle C that contains u and w. Now since G is a block, w is not a cut point of G and so $G - w$ is. Hence there exists a $u - v$ path P not containing w. Let v' be the last point common to P and C. (See Fig.1.27). Since$u$ is common to P and C, such a v' exists. Now, let Q denote the $u - v'$ path along the cycle C not containing the point w.Then, Q followed by the $v' - v$ path along P, the line vw and the $w - v$ path along the cycle that contains both u and v. This completes the induction.

Thus any two points of G lie on a common cycle of G.

(2) $\Rightarrow$ (1).Suppose any two points of G lie on a common cycle of G. Suppose v is a cut point of G. Then there exists two points u and w distinct from v such that every $u - w$ path contains v.(Refer Theorem 4.8). Now, by hypothesis $u$and w lie on a common cycle and this cycle determines two $u - w$ paths and at least one of these paths does not contain v which is a contradiction. Hence G has no cut points so that G is a block.

(2) $\Rightarrow$ (3). Let u be a point and vw a line of G. By hypothesis u and v lie on a common cycle C. If w lies on C, then the line uw together with the $v - w$ path of C containing u is the required cycle containing u and the line vw. If w is not on C, let C' be a cycle containing u and w. This cycles determines two $w - u$ paths and at least one of these paths does not contain v. Denote this path by P. Let u' be the first point common to P and C.(u' may be u itself). Then the line vw followed by the $w - u'$ sub path of P and the $u' - v$ path in C containing u form a cycle containing u and the line vw. (3) $\Rightarrow$ (2) is trivial.

(3) $\Rightarrow$ (4). The proof is analogous to the proof of (2) $\Rightarrow$ (3) and is left as an exercise. (4) $\Rightarrow$ (3) is trivial. $\square$

1.24 Exercise

1. Prove that each line of a graph lies in exactly one of its blocks.
2. Prove that the lines of any cycle of G lie entirely in a single block of G
3. Prove that if a point v is common to two distinct block of$G$, then v is a cut point of G.
4. Prove that a graph G is a block iff for any three distinct points of G, there is a path joining any two of them which does not contain the third.
5. Prove that a graph G is a block iff for any three distinct points of G, there is a path joining any two of them which contains the third.

1.25 Connectivity

We define two parameters of a graph, its connectivity and edge connectivity which measures the extend to which it is connected.

Definition 1.25.1. The **connectivity** $\kappa = \kappa(G)$ of a graph G is the minimum number of points whose removal results in a disconnected or trivial graph. The connectivity $\lambda = \lambda(G)$ of G is the minimum number of lines whose removal results in a disconnected or trivial graph.

Example 1.25.1.

1. The connectivity and line connectivity of a disconnected graph is 0.
2. The connectivity of a connected graph with a cut point is 1.
3. The line connectivity of a connected graph with a bridge is 1.
4. The complete graph K_p cannot be disconnected by removing any number of points, but the removal of $p-1$ points results in a trivial graph.Hence $\kappa(K_p) = p - 1$

Theorem 1.25.1. For any graph G ,$\kappa \leq \lambda \leq \delta$.

Proof. We first prove $\lambda \leq \delta$. If G has no lines, $\lambda = \delta = 0$. Other wise removal of all the lines incident with a point of minimum degree results in a disconnected graph . Hence $\lambda \leq \delta$. Now to prove $\kappa \leq \lambda$, we consider the following cases.

Case(i)G is disconnected or trivial.Then $\kappa = \lambda = 0$

Case(ii)G is a connected graph with a bridge x. Then $\lambda = 1$.Further in case $G = K_2$ or one of the points incident with$x$ is a cut point. Hence $\kappa = 1$ so that $\kappa = \lambda = 1$.

Case(iii)$\lambda \geq 2$.Then there exist λ lines the removal of which disconnects graph. hence the removal of $\lambda - 1$ of lines results in a graph G with bridge $x = uv$. For each of these $\lambda - 1$ line select an incident point different from u or v .The removal of these $\lambda - 1$ points removes all the $\lambda - 1$ lines. If the resulting graph is disconnected, then $\kappa \leq \lambda - 1$.If not x is a bridge of this subgraph and hence the removal of u or v results in a disconnected or trivial graph. Hence $\kappa \leq \lambda$ and this completes the proof. □

Remark 1.25.1. The inequalities in theorem 1.25.1 are often strict. For the graph given in fig 1.28 $\kappa = 2, \lambda = 3$ and $\delta = 4$.

Definition 1.25.2. A graph G is said to be **n-connected** if $\kappa(G) \geq n$ and **n-line connected** if $\lambda(G) \geq n$.

Thus a non trivial graph is $1-$ connected iff it is connected. A non trivial graph is $2-$ connected iff it is block having more than one line. Hence K_2 is the only block which is not $2-$ connected.

1.26 Solved Problems

Problem 10. Prove that G is $k-$ connected graph then $q \geq \frac{pk}{2}$.

Solution. Since G is $k-$ connected, $k \leq \delta$(by theorem 1.25.1).

$$\begin{aligned} \therefore q &= \frac{1}{2} \\ &\geq \frac{1}{2}p\delta \text{ (since } d(v) \geq \delta \text{ for all } v \\ &\geq \frac{pk}{2}. \end{aligned}$$

Problem 11. Prove that there is no $3-$ connected graph with 7 edges.
Solution Suppose G is a $3-$ connected graph with 7 edges.G has 7 edges $\Rightarrow p \geq 5$. Now $q \geq \frac{3p}{2}$. Therefore $q \geq \frac{15}{2}$. Hence $q \geq 8$ which is a contradiction. Hence there is no $3-$ connected graph with 7 edges.

1.27 Exercise

1. Find the connectivity of $K_{m,n}$.
2. Show that if G is $n-$ line connected and$E$ is a set of n lines,the the number of components in the graph $G - E$ is either 1 or 2.
3. give an example to show that the analogue of the above result is not true for a $n-$ connected graph.
4. Give an example of a closed walk of even length which does not contain a cycle.
5. Give an example to show that the union of two distinct $u - v$ walks need not contain a cycle.
6. Prove that the union of two distinct $u - v$ paths contain a cycle.
7. Show that if a line is in a closed trail of G then it is in a cycle of G.
8. Determine which of the following statements are true and which are false.
   (a) Any $u - v$ walk contains a $u - v$ path.
   (b) The union of any two distinct $u - v$ walks contains a cycle.
   (c) The union of any two distinct $u - v$ paths contains a cycle.
   (d) A graph is connected iff it has only one component.
   (e) The complement of a connected graph is connected
   (f) Any subgraph of a connected graph is connected
   (g) An induced subgraph of a connected graph is connected
   (h) If a graph has a cut point ,then it has a bridge.
   (i) If a graph has a bridge ,then it has a cut point.

(j) If v is a cut point of a G then $\omega(G-v) = \omega(G) + 1$

(k) If x is a bridge of G,then $\omega(G-x) = \omega(G) + 1$

(l) In a connected graph every line can be a bridge.

(m) In a connected graph every point can be a cut point.

(n) A point common to two distinct blocks of a graph G is a cut point of G.

(o) Every line of a graph G lies in exactly one block of G.

(p) If a graph is $n-$ connected then it is $n-$ line connected.

(q) Every block is $2-$ connected.

Answers

1,3,4,11,12,14,15 and 16 are true.

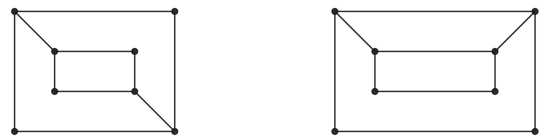

Figure 1.17:

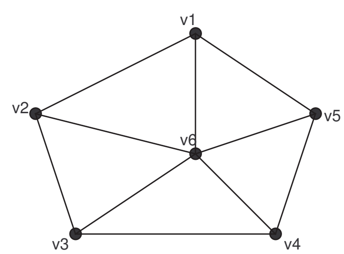

Figure 1.18:

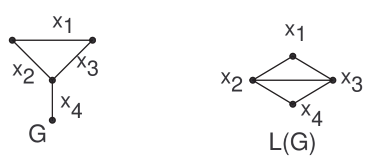

Figure 1.19:

Figure 1.20:

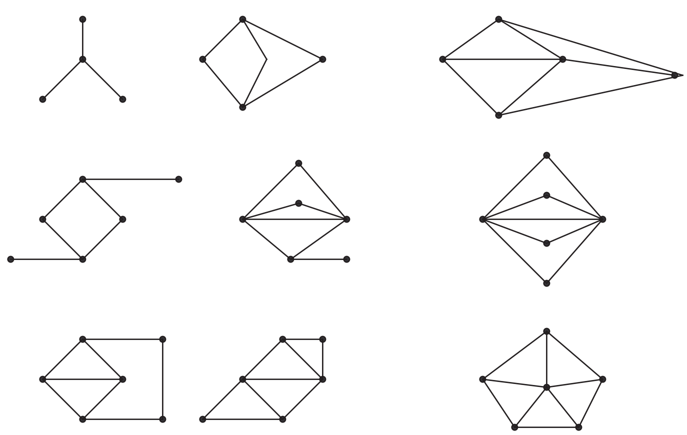

Figure 1.21:

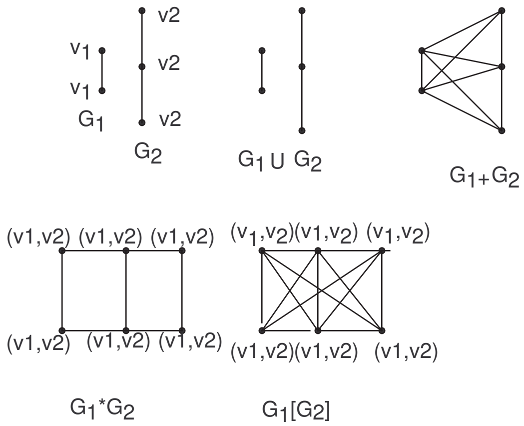

Figure 1.22:

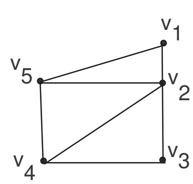

Figure 1.23:

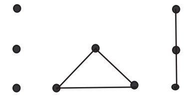

Figure 1.24:

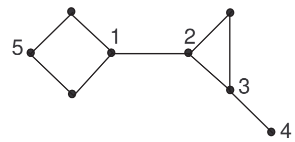

Figure 1.25:

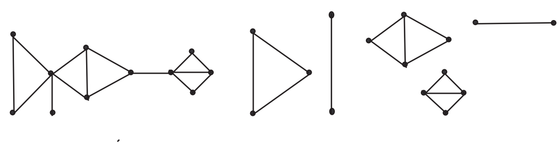

Figure 1.26:

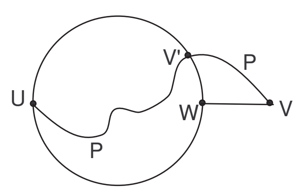

Figure 1.27:

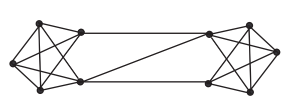

Figure 1.28:

Module 2

Eulerian graphs, Hamiltonian graphs and Trees

2.1 Eulerian graphs

Definition 2.1.1. A closed trail containing all the points and lines is called an eulerian trail. A graph having an eulerian trail is called an eulerian graph.

Remark 2.1.1. In an eulerian graph, for every pair of points u and v there exists at least two edge disjoint $u-v$ trails and consequently there are at least two edge disjoint $u-v$ paths. The graph shown in figure 2.1 is eulerian.

Theorem 2.1.1. If G is a graph in which the degree of every vertex is at least two then G contains a cycle.

Proof. First, we construct a sequence of verices $v_1, v_2, v_3, \ldots$ as follows. Choose any vertex v. Let v_1 be any vertex adjacent to v. Let v_2 be any vertex adjacent

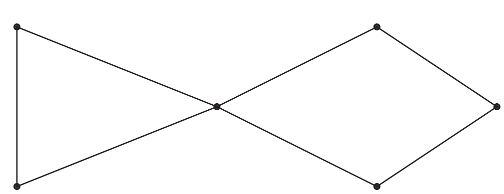

Figure 2.1: A Eulerian graph

to v_1 other than v. At any stage, if the vertex v_i, $i \geq 2$ is already chosen, then choose v_{i+1} to be any vertex adjacent to v_i other than v_{i-1}. Since degree of each vertex is at least 2, the existence of v_{i+1} is always guaranteed. G has only finite number of vertices, at some stage we have to choose a vertex which has been chosen before. Let v_k be the first such vertex and let $v_k = v_i$ where $i < k$. Then $v_i v_{i+1} \dots v_k$ is a cycle. □

Theorem 2.1.2. Let G be a connected graph. Then the following statements are equivalent.

(1) G is eulerian.

(2) every point has even degree.

(3) the set of edges of G can be partitioned into cycles.

Proof.

(1) ⇒ (2) Assume that G is eulerian. Let T be an eulerian trail in G, with origin and terminus u. Each time a vertex v occurs in T in a place other than the origin and terminus, two of the edges incident with v are accounted for. Since an eulerian trail contains every edges of G, $d(v)$ is even for $v \neq u$. For u, one of the edges incident with u is accounted for by the origin of T, another by the terminus of T and others are accounted for in pairs. Hence $d(u)$ is also even.

(2) ⇒ (3) Since G is connected and nontrivial every vertex of G has degree at least 2. Hence G contains a cycle Z. The removal of the lines of Z results in a spanning subgraph G_1 in which again vertex has even degree. If G_1 has no edges, then all the lines of G form one cycle and hence (3) holds. Otherwise, G_1 has a cycle Z_1. Removal of the lines of Z_1 from G_1 results in spanning subgraph G_2 in which every vertex has even degree. Continuing the above process, when a graph G_n with no edge is obtained, we obtain a partition of the edges of G into n cycles.

(3) ⇒ (1) If the partition has only one cycle, then G is obviously eulerian, since it is connected. Otherwise let $z_1, z_2, \dots, z_n$ be the cycles forming a partition of the lines of G. Since G is connected there exists a cycle $z_i \neq z_1$ having a common point v_1 with z_1. Without loss of generality,

let it be z_2. The walk beginning at v_1 and consisting of the cycles z_1 and z_2 in succession is a closed trail containing the edges of these two cycles. Continuing this process, we can construct a closed trail containing all the edges of G. Hence G is eulerian.

$\square$

Corollary 2.1.1. Let G be a connected graph with exactly $2n(n \geq 1)$, odd vertices. Then the edge set of G can be partitioned into n open trails.

Proof. Let the odd vertices of G be labelled $v_1, v_2, \ldots, v_n; w_1, w_2, \ldots, w_n$ in any arbitrary order. Add n edges to G between the vertex pairs $(v_1, w_1), (v_2, w_2), \ldots, (v_n, w_n)$ to form a new graph G'. No two of these n edges are incident with the same vertex. Further every vertex of $G'$is of even degree and hence G' has an eulerian trail T. If the n edges that we added to G are now removed from T, it will split into n open trails. These are open trails in G and form a partition of the edges of G. $\square$

Corollary 2.1.2. Let G be a connected graph with exactly two odd vertices. Then G has an open trail containing all the vertices and edges of G.

Corollary 2.1.2 answers the question: Which diagrams can be drawn without lifting one's pen from the paper not covering any line segment more than once?

Definition 2.1.2. A graph is said to be arbitrarily traversable(traceable)from a vertex v if the following procedure always results in an eulerian trail. Start at v by traversing any incident edge. On arriving at a vertex u, depart through any incident edge not yet traversed and continue until all the lines are traversed.
If a graph is arbitrary traversable from a vertex then it obviously eulerian.

The graph shown in figure 2.1 is arbitrarily traversable from v. From no other point it is arbitrarily traversable.

Theorem 2.1.3. An eulerian graph G is arbitrarily traversable from a vertex v in G iff every cycle in G contains v.

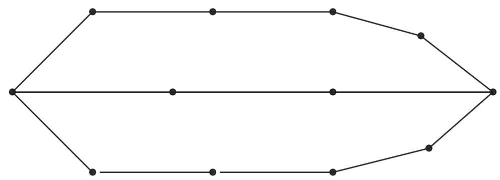

Figure 2.2: A theta graph

2.1.1 Exercise

1. For what values of n, is K_n eulerian?

2. For what values of m and n is $K_{n,m}$ is eulerian?

3. Show that if G has no vertices of odd degree, then there are edge disjoint cycles $C_1, C_2, \ldots, C_n$ such that

$$E(G) = E(C_1) \cup E(C_2) \cup \ldots \cup E(C_m)$$

4. Show that every block of a connected graph G is eulerian then G is eulerian.

2.2 Hamiltonian Graphs

Definition 2.2.1. A spanning cycle in a graph is called a hamiltonian cycle. A graph having a hamiltonian cycle is called a hamiltonian graph.

Definition 2.2.2. A block with two adjacent vertices of degree 3 and all other vertices of degree 2 is called a theta graph.

Example 2.2.1. The graph shown in figure 2.2is a theta graph. A theta graph is obviously nonhamiltonian and every nonhamiltonian 2-connected graph has a theta subgraph.

Theorem 2.2.1. Every hamiltonian graph is 2-connected.

Proof. Let G be a hamiltonian graph and let Z be a hamiltonian cycle in G. For any vertex v of G, $Z - v$ is connected and hence $G - v$ is also connected. Hence G has no cutpoints and thus G is 2-connected. □

Theorem 2.2.2. If G is hamiltonian, then for every nonempty proper subset S of $V(G)$, $\omega(G-S) \leq |S|$ where $\omega(H)$ denote the number of components in any graph H.

Proof. Let Z be a hamiltonian cycle of G. Let S be any nonempty proper subset of $V(G)$. Now, $\omega(Z-S) \leq |S|$. Also $Z-S$ is a spanning subgraph of $G-S$ and hence $\omega(G-S) \leq \omega(Z-S)$. Hence $\omega(G-S) \leq |S|$. □

Theorem 2.2.3. The bipartite graph $K_{m,n}$ is nonhamiltonian.

Proof. Let (V_1, V_2) be a bipartition of the graph with $|V_1| = m$ and $|V_2| = n$. The graph $K_{m,n} - V_1$ is the totally disconnected graph with n points. Hence $\omega(K_{m,n} - V_1) = n > m = |V_1|$. Therefore $K_{m,n}$ is non hamiltonian. □

Remark 2.2.1. The converse of theorem 2.2.2 is not true. For example, Petersen graph satisfies the condition of the theorem but is nonhamiltonian.

Theorem 2.2.4. If G is a graph with $p \geq 3$ vertices and $\delta \geq p/2$, then G is hamiltonian.

Proof. Suppose the theorem is false. Let G be a maximal nonhamiltonian graph with p vertices and $\delta \geq p/2$. Since $p \geq 3$, G can not be complete. Let u and v be nonadjacent vertices in G. By the choice of G, $G + uv$ is hamiltonian. Moreover, since G is nonhamiltonian, each hamiltonian cycle of $G + uv$ must contain the line uv. Thus G has a spanning path $v_1, v_2, \ldots, v_p$ with origin $u = v_1$ and terminus $v = v_p$. Let $S = \{v_i : uv_{i+1} \in E\}$ and $T = \{v_i : i < p \text{ and } v_i v \in E\}$ where E is the edge set of G. Clearly $v_p \notin S \cup T$ and hence

$$|S \cup T| < p \tag{2.1}$$

Again if $v_i \in S \cap T$, then $v_1 v_2 \ldots v_i v_p v_{p-1} \ldots v_{i+1} v_i$ is a hamiltonian cycle in G, contrary to the assumption. Hence $S \cap T = \emptyset$ so that

$$|S \cap T| = 0. \tag{2.2}$$

Also by the definition of S and T, $d(u) = |S|$ and $d(v) = |T|$. Hence by equations (2.1) and (2.2) , $d(u) + d(v) = |S| + |T| = |S \cup T| < p$. Thus $d(u) + d(v) < p$. But since $\delta \geq p/2$, we have $d(u) + d(v) \geq p$ which gives a contradiction. □

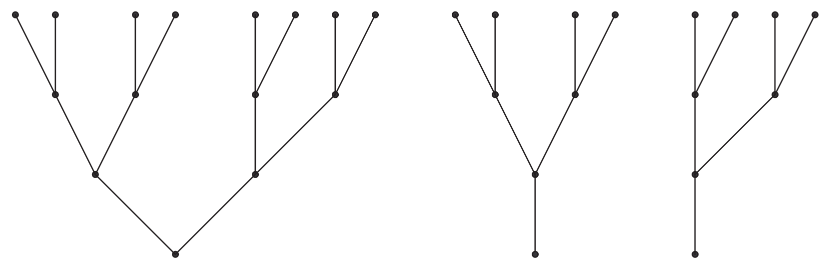

Figure 2.3: A tree(left) and a forest(right)

Lemma 1. Let G be a graph with p points and let u and v be nonadjacent points in G such that $d(u) + d(v) \geq p$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.

Proof. First, assume that G is hamiltonian. Then obviously $G + uv$ is hamiltonian. Conversely, assume that $G + uv$ is hamiltonian, but G is not. Then, as in the proof of theorem 2.2.4, we obtain $d(u) + d(v) < p$. This contradicts the hypothesis that $d(u) + d(v) \geq p$. Thus $G + uv$ is hamiltonian implies G is hamiltonian. □

2.3 Trees

2.3.1 Characterization of Trees

Definition 2.3.1. A graph that contains no cycles is called a an acyclic graph. A connected acyclic graph is called a tree.A graph without cycles is also called a forest so that the components of a forest are trees.

Example 2.3.1. An example of a tree and a forest is shown in figure 2.3.

Theorem 2.3.1. Let G be a (p, q) graph. The following statements are equivalent.

(1) G is a tree.

(2) every two points of G are joined by a unique path.

(3) G is connected and $p = q + 1$

(4) G is acyclic and $p = q + 1$

Proof.

(1) $\Rightarrow$ (2) Assume that G is a tree. Let u and v be any two points of G. Since G is connected there exists a $u - v$ path in G. Now suppose that there exists two distinct $u - v$ paths, say:

$$P_1 : u = v_0, v_1, v_2, \ldots, v_n = v \text{ and } P_2 : u = w_0, w_1, \ldots, w_m = v$$

Let i be the least positive integer such that $1 \leq i < m$ and $w_i \notin P_1$ (such an i exists since P_1 and P_2 are distinct). Hence $w_{i-1} \in P_1 \cap P_2$. Let j be the least positive integers such that $i < j \leq m$ and $w_j \in P_1$. Then the $w_{i-1} - w_j$ path along P_2 followed by the $w_j w_{i-1}$ path along P_1 form a cycle which is a contradiction. Hence there exists a unique $u - v$ path in G.

(2) $\Rightarrow$ (3) Assume that every two points of G are joined by a unique path. This implies that G is connected. We will show that $p = q+1$ by induction on p. The result is trivial for connected graphs with 1 or 2 points. Assume that the result is true for all graphs with fewer than p points. Let G be a graph with p points. Let $x = uv$ be any line in G. Since there exists a unique $u - v$ path in G, $G - x$ is a disconnected graph with exactly two components G_1 and G_2. Let G_1 be a (p_1, q_1) graph and G_2 be a (p_2, q_2) graph. Then $p_1 + p_2 = p$ and $q_1 + q_2 = q - 1$. Further by induction hypothesis $p_1 = q_1 + 1$ and $p_2 = q_2 + 1$. Hence

$$p = p_1 + p_2 = q_1 + q_2 + 2 = q - 1 + 2 = q + 1$$

(3) $\Rightarrow$ (4) Assume that G is connected and $p = q + 1$. We will show that G is acyclic. Suppose G contains a cycle of length n. There are n points and n lines on this cycle. Fix a point u on the cycle. Consider any one the remaining $p - n$ points not on the cycle, say v. Since G is connected we can find a shortest $u - v$ path in G. Consider the line on this shortest path incident with v. The $p - n$ lines thus obtained are all distinct. Hence $q \geq (p-n)+n = p$ which is a contradiction since $q+1 = p$. Thus G is acyclic.

(4) $\Rightarrow$ (1) Assume that$G$ is acyclic and $p = q + 1$. We will prove that G is a tree. Since G is acyclic to prove that G is a tree we need only prove that G is connected. Suppose G is not connected. Let $G_1, G_2, \ldots, G_k (k \geq 2)$ be the components of G. Since G is acyclic each of these components is a tree. Thus $q_i + 1 = p_i$ where G_i is a (p_i, q_i) graph. This implies that $\sum_{i=1}^{k} q_i + 1) = \sum_{i=1}^{k} p_i$. That is, $q + k = p$ and $k \geq 2$, which is a contradiction. Hence G is connected.

$\square$

Corollary 2.3.1. Every non trivial tree G has at least two vertices of degree one.

Proof. Since G is non trivial, $d(v) \geq 1$ for all points v. Also $\sum d(v) = 2q = 2(p-1) = 2p - 2$. Hence $d(v) = 1$ for at least two vertices. $\square$

Theorem 2.3.2. Every connected graph has a spanning tree.

Proof. Let G be a connected graph. Let T be a minimal connected spanning subgraph of G. Then for any line x of T, $T - x$ is disconnected and hence x is a bridge of T. Hence T is acyclic. Further T is connected and hence is a tree. $\square$

Corollary 2.3.2. Let G be a (p, q) connected graph. Then $q \geq p - 1$.

Proof. Let T be a spanning tree of G. Then the number of lines in T is $p - 1$. Hence $q \geq p - 1$.

$\square$

Theorem 2.3.3. Let T be a spanning tree of a connected graph G. Let $x = uv$ be an edge of G not in T. Then $T + x$ contains a unique cycle.

Proof. Since T is acyclic every cycle in $T + x$ must contain x. Hence there exists a one to one correspondence between cycles in $T + x$ and $u - v$ paths in T. As there is a unique $u - v$ path in tree T, there is a unique cycle in $T + x$. $\square$

2.3.2 Centre of a Tree

Definition 2.3.2. Let v be a point in a connected graph G. The eccentricity $e(v)$ of v is defined by $e(v) = \max\{d(u, v) : u \in V(G)\}$. The radius $r(G)$ is defined by $r(G) = \min\{e(v) : v \in V(G)\}$. The point v is called the central point if $e(v) = r(G)$ and the set of central points is called the centre of G.

Theorem 2.3.4. Every tree has a centre consisting of either one point or two adjacent points.

Proof. The result is trivial if $G = K_1$ or K_2. So assume that let T be any tree with $p \geq 2$ points. T has at least two end points and maximum distance from a given point u to any other point v occurs only when v is an end point. Now delete all the end points from T. The resulting graph T' is also a tree and eccentricity of each point in T' is exactly one less than the eccentricity of the same point in T. Hence T and T' have the same centre. If the process of removing the end points is repeated, we obtain successive trees having the same centres as T and we eventually obtain a tree which is either K_1 or K_2. Hence the centre of T consists of either one point or two adjacent points.

$\square$

2.3.3 Exercise

1. Show that there does not exists a nonhamiltonian graph with arbitrarily high eccentricity.

2. Prove that a graph G is tree iff G is connected and every line of G is a bridge.

3. Prove that if G is a forest with p points and k components then G has $p - k$ lines.

4. Prove that the origin and terminus of a longest path in a tree have degree one.

5. Show that every tree with exactly 2 vertices of degree one is a path.

6. Show that every tree is a bipartite graph. Which trees are complete bipartite graphs.

7. Prove that every block of a tree is K_2.
8. Draw all trees with 4 and 5 vertices.
9. Prove that any edge of a connected graph G one of whose end point is of degree one is contained in every spanning tree of G.
10. Prove that a line x of a connected graph is in every spanning tree of G iff x is a bridge.

Module 3

Matchings and Planarity

3.1 Matchings

Definition 3.1.1. Any set M of independent lines of a graph G is called a *matching* of G. If $uv \in M$, we say that u and v are matched under M. We also say that the points u and v are M-saturated. A matching M is called a perfect matching if every point of G is M-saturated. M is called a maximum matching if there is no matching M' in G such that $|M'| > |M|$.

Example 3.1.1. Consider the graph G_1 shown in figure 3.1. Let $M_1 = \{v_1v_2, v_6v_3, v_5v_4\}$ is a perfect matching in G_1. Also $M_2 = \{v_1v_3, v_6v_5\}$ is a matching in G_1. However M_2 is not a perfect matching. The points v_2 and v_4 are not M_2 saturated. For the graph G_2, $M_2 = \{v_8v_4, v_1v_2\}$ is a maximum matching but not a perfect matching.

Definition 3.1.2. Let M be a matching in $G = (V, E)$. A path in G is called an M-alternating path if its lines are alternatively in $E - M$ and M. An M-alternating path whose origin and terminus are both M-unsaturated is called an M-augmenting path.

Example 3.1.2. Consider the graph G_1 shown in figure in 3.1. $P_1 = \{v_6, v_5, v_4, v_3\}$ is an M_1 alternating path. Also $P_2 = \{v_2, v_1, v_3, v_6, v_5, v_4\}$ is an M_2 augmented path. In the graph G_2, (v_7, v_9, v_4) is an M-alternating path.

Remark 3.1.1. If a graph G has a perfect matching M, then $p = 2|M|$ and hence p is even. However the converse is not true. The graph G_2 shown in figure 3.1 has an even number of vertices but has no perfect matching.

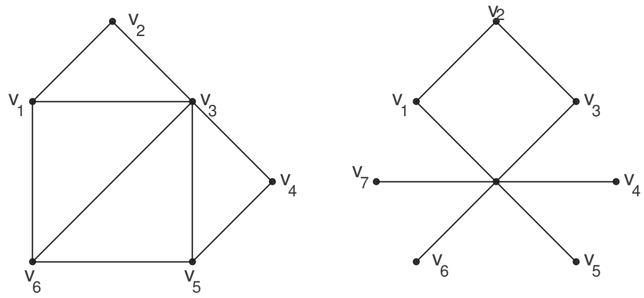

Figure 3.1: Graphs G_1(left) and G_2(right)

Theorem 3.1.1. Let M_1 and M_2 be two matchings in a graph G. Let $M_1 \triangle M_2$ be the symmetric difference of M_1 and M_2. Let $H = G[M_1 \triangle M_2]$ be the subgraph of G induced by $M_1 \triangle M_2$. Then each component of H is either an even cycle with edges alternatively in M_1 and M_2 or a path P with edges alternatively in M_1 and M_2 such that the origin and the terminus of P are unsaturated in M_1 or M_2.

Proof. Let v be any point in H. Since M_1 and M_2 are matchings in G, at most one line of M_1 and at most one line of M_2 are incident with v. Hence the degree of v in H is either 1 or 2. Hence it follows that the components of H must be as described in the theorem. $\square$

Example 3.1.3. Consider the graph G_1 shown in figure 3.1. Note that

$$M_1 \triangle M_2 = \{v_1v_2, v_6v_3, v_5v_4, v_1v_3, v_6v_5\}$$

The graph $H_1 = G_1[M_1 \triangle M_2]$ is shown in figure 3.2.

Clearly H_1 is a path whose edges are alternatively in M_1 or M_2. The origin v_2 and the terminus v_4 are both M_2 - unsaturated. The following theorem due to Berge gives a characterization of maximum matching.

Theorem 3.1.2. A matching M in a graph G is a maximum matching if and only if G contains no M-augmented path.

Proof. Let M be a maximum matching in G. Suppose G contains an M-augmented path $P = (v_0, v_1, \ldots, v_{2k+1})$. By the definition of M-augmenting

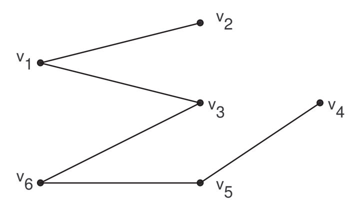

Figure 3.2: The graph H_1

path the lines $v_0v_1, v_2v_3, \ldots, v_{2k}v_{2k+1}$ are not in M and the lines $v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}$ are in M. Hence

$$M' = M - \{v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}\} \cup \{v_0v_1, v_2v_3, \ldots, v_{2k}v_{2k+1}\}$$

is a matching in G and $|M'| = |M| + 1$, which is a contradiction, since M is a maximum matching. Hence G also has no M-augmenting path.

Conversely, suppose G has no M-augmenting path. If M is a not a maximum matching in G then there is exits a matching M' of G such that $|M'| > |M|$. Let $H = G[M \triangle M']$. By theorem 3.1.1, each component of H is either an even cycle with edges alternatively in M and M' or a path P with edges alternatively in M and M' such that the origin and terminus of P are unsaturated in M or M'. Clearly any component of H which is a cycle contains equal number of edges from M and M'. Since $|M'| > |M|$ there exists at least one component of H which is a path whose first and last edges are from M'. Thus the origin and terminus of P are M'- unsaturated in H and hence they are M-saturated in G. Thus P is an M-augmenting path in G, which is a contradiction. Hence M is maximum matching in G. □

3.2 Worked Problems

Problem 12. For what values of n does the complete graph K_n have perfect matching.

Clearly any graph with p odd has no perfect matching. Also the complete graph K_n has a perfect matching if n is even. For example, if $V(K_n) = \{1, 2, \ldots, n\}$ then $\{12, 34, \ldots, (n-1)n\}$ is a perfect matching of K_n. Thus K_n

has a perfect matching if and only if n is even.

Problem 13. Show that a tree has at most one perfect matching.

Let T be a tree. Suppose T has two perfect matchings say M_1 and M_2. Then degree of every vertex in $H = T[M_1 \triangle M_2]$ is 2. Hence every component of H is an even cycle with edges alternatively in M_1 and M_2. This is a contradiction, since T has no cycles. Therefore T has at most one perfect matching.

Problem 14. Find the number of perfect matching in the complete bipartite graph $K_{n,n}$.

Let $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_n\}$ be a bi-partition of $K_{n,n}$. We observe that any matching of $K_{n,n}$ that saturates every vertex of A is a perfect matching. Now the vertex x_1 can be saturated in n ways by choosing any one of the edges $x_1y_1, x_1y_2, \ldots, x_1y_n$. Having saturated x_1 the vertex x_2 can be saturated in $n-1$ ways. In general having saturated $x_1, x_2, \ldots, x_i$ the next vertex x_{i+1} can be saturated in $n-i$ ways. Hence the number of perfect matchings in $K_{n,n}$ is $n.(n-1)\ldots 2.1 = n!$.

Problem 15. Find the number of perfect matchings in the complete graph K_{2n}.

Let $V(K_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$. The vertex v_1 can be saturated in $2n-1$ way by choosing any line e_1 incident at v_1. In general having chosen $e_1, e_2 \ldots, e_k$ can be saturated in $2n-(2k+1)$ ways. We obtain a perfect matching after the choice of n lines in the above process. Hence the number of perfect matching in K_{2n} is equal to $1.3.5\ldots(2n-1)$. Note that

$$\begin{aligned} 1.3.5\ldots(2n-1) &= \frac{1.2.3.4.5.\ldots.(2n-1)(2n)}{2.4.6\ldots 2n} \\ &= \frac{(2n)!}{2^n n!} \end{aligned}$$

3.3 Exercise

1. Find maximum matching in the tree shown in figure 3.3.

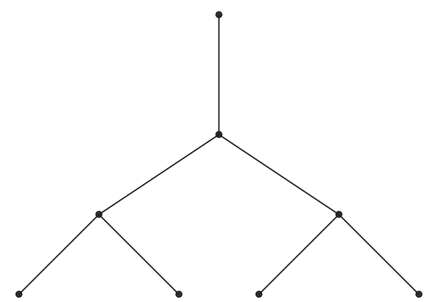

Figure 3.3: A Tree

2. Prove that a 2-regular graph G has a perfect matching if and only if every component of G is an even cycle.

3. Give an example of a 3- regular graph which has no perfect matching.

3.4 Matchings in Bipartite Graphs

3.4.1 Personnel Assignment Problem

In a company, n workers $x_1, x_2, \ldots, x_n$ and m jobs $j_1, j_2, \ldots, j_m$ are available. Each worker is qualified for at least one of the jobs. Is it possible to assign one job for each worker for which he is qualified? This problem is known as personnel assignment problem. We construct a bipartite graph G with bipartition $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{j_1, j_2, \ldots, j_m\}$. x_i being joined to j_k if and only if x_i is qualified for the job j_k. The personnel assignment problem reduce to the following question. Does G have a matching that saturates every vertex in A?

3.4.2 The marriage Problem

Let $A = \{x_1, x_2, \ldots, x_n\}$ be a set of n boys and $B = \{y_1, y_2, \ldots, y_m\}$ be a set of m girls in a village. Each boy has one or more girl friends. Under what conditions can we arrange marriage in such a way that each boy marries one of his girl friends? This problem is known as the marriage problem.

We now obtain a graph theoretical formulation of the above problem. Let

G be the bipartite graph with the bi-partition $\{A, B\}$ such that x_i is joined to y_j if and only if y_i is a girl friend of x_i. The marriage is equivalent to finding the conditions under which G has a matching that saturates every vertex of A.

Definition 3.4.1. For a subset S of V the neighbor set $N(S)$ is the set of all points each of which is adjacent to at least one vertex in S.

Theorem 3.4.1 (Halli's Marriage Theorem)**.** Let G be a bipartite graph with bi-partition (A, B). Then G has a matching that saturates all the vertices of A if and only if $|N(S)| \geq |S|$, for every subset S of A.

Proof. Suppose G has a matching M that saturates all the vertices in A.Let $S \subseteq A$. Then every vertices in S is matched under M to a vertex in $N(S)$ and two distinct vertices of $N(S)$. Hence it follows that $|N(S)| \geq |S|$.

Conversely, suppose $|N(S)| \geq |S|$ for all $S \subseteq A$. We wish to show that G contains a matching which saturates every vertex in A. Suppose G has no such matching. Let M^* be a maximum matching in G. By assumption there exists a vertex $x_0 \in A$ which is M^* unsaturated. Let

$$Z = \{v \in V(G) : \text{there exists a } M^* \text{ alternating path conecting } x_0 \text{ and } v\}$$

Since M^* is a maximum matching, by Berge's theorem, G has no no M^* augmenting path and hence x_0 is the only M^* unsaturated vertex in Z. Let $S = Z \cap A$ and $T = z \cap B$. Clearly x_0 in S and every vertex of $S - \{x_0\}$ is matched under M^* with a vertex of T. Thus

$$|T| = |S| - 1. \tag{3.1}$$

We now claim that $N(S) = T$. Clearly, from the definition of T, we have

$$T \subseteq N(S). \tag{3.2}$$

Now let $v \in N(S)$. Hence there exists $u \in S$ such that v is adjacent to u. Since $S = Z \cap A$ it follows that $u \in Z$. Hence there exits an M^* alternating path $P = (x_0, y_1, x_1, y_2, \ldots, x_{k-1}, y_k, u)$. If v lies on P, then clearly $v \in Z \cap B = T$. Suppose v does not lie on P. Now the edge $y_k u \in M^*$. Hence the edge uv is not in M^*. Hence the path P_1 consisting of P followed by the edge uv is an M^*-alternating path. Hence $v \in Z \cap B = T$. Thus

$$N(S) \subseteq T. \tag{3.3}$$

From equations (3.2) and (3.4.2) we have

$$N(S) = T \tag{3.4}$$

From equations (3.1) and (3.4) we have

$$|N(S)| = |T| = |S| - 1 < |S|$$

which is a contradiction. □

Remark 3.4.1. Hall's theorem answers the marriage problem. The marriage problem with n boys has a solution if and only if for every k with $1 \leq k \leq n$, every set of k boys has collectively at least k girl friends.

The following is an important consequence of Hall's marriage theorem.

Theorem 3.4.2. Let G be a k regular bipartite graph with $k > 0$. Then G has a perfect matching.

Proof. Let (V_1, V_2) be a bi-partition of G. Since each edge of G has one end in V_1 and there are k edges incident with each vertex of V_1, we have $q = k|V_1|$. By a similar argument $q = k|V_2|$, so that $k|V_1| = k|V_2|$. Since $k > 0$, we get $|V_1| = |V_2|$. Now let $S \leq V_1$. Let E_1 denote the set of all edges incident with vertices in $N(S)$. Since G is k- regular, $|E_1| = k|E_2|$ and $|E_2| = k|N(S)|$. Also by definition of $N(S)$, we have $E_1 \subseteq E_2$, and hence it follows that $k|S| \leq k|N(S)|$. Thus $|N(S)| \geq |S|$. Hence by Hall's theorem, G, has a matching M that saturates every vertex in V_1. Since $|V_1| = |V_2|$, M also saturates all the vertices of V_2. Thus M is a perfect matching.

□

3.5 Exercise

1. For any graph G, let $O(G)$ denote the number of odd components of G. Let $G = (V, X)$ be any graph. Prove that if G has a perfect matching

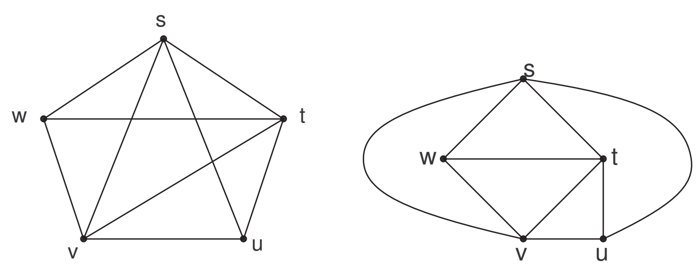

Figure 3.4: A planar graph (left) and its embbeding (right)

M, then $O(G - S) \leq |S|$ for all $S \subseteq V$.

2. Using the above problem show that the following graph has no perfect matching.

3.6 Planarity

Definition 3.6.1. A graph is said to be embedded in a surface S when it is drawn on S such that no two edges intersect(meetins of edges at a vertex is not considered an intersection). A graph is called planar if it can be drawn on a plane without intersecting edges. A graph is called non planar if it is not planar. A graph that is drawn on the plane without intersecting edges is called a plane graph.

Example 3.6.1. The graph shown in figure (3.4) is planar.

Theorem 3.6.1. The complete graph K_5 is non planar.

Proof. If possible, let K_5 be planar. Then K_5 contains a cycle of length 5 say (s, t, u, v, w, s). Hence, without loss of generality, any plane embedding of K_5 can be assumed to contain this cycle drawn in the form of a regular pentagon. Hence the edge wt must lie either wholly inside the pentagon or wholly outside it.

Suppose that wt is wholly inside the pentagon(the argument when it lies wholly outside the pentagon is quite similar). Since the edge sv and su do not cross the edge wt, they must be both lie outside the pentagon. The edge vt cannot cross the edge su. Hence vt must be inside the pentagon. But now, the edge uw crosses one of the edges already drawn, giving a contradiction. Hence K_5 is not planar. □

Definition 3.6.2. Let G be a graph embedded on a plane π. Then $\pi - G$ is the union of disjoint regions. Such regions are called faces of G. each plane graph has exactly one unbounded face and it is called the exterior face. Let F be a face of plane graph G and e be an edge of G. Let P be a point in F. e is said to be in the boundary of F if for every point Q of π on e there exists a curve joining P and Q which lies entirely in F.

Theorem 3.6.2. A graph can be embedded in the surface of a sphere iff it can be embedded in a plane.

Proof. Let G be a graph embedded on a sphere. Place the sphere on the plane L and call the point of contact S(south pole). At point S, draw a normal to the plane and let N (North pole) be the point where this normal intersects the surface of the sphere.

Assume that the sphere is placed in such a way that N is disjoint from G. For each point P on the sphere, let P' be the unique point on the plane where the line NP intersects the surface of the plane. There is a one to one correspondence between the points of the sphere other than N and the points on the plane. In this way, the vertices and the edges of G can be projected on the plane L, which gives an embedding of G in L.

The reverse process obviously gives an embedding in the sphere for any graph that is embedded in the plane L. This completes the proof.

□

Theorem 3.6.3. Every planar graph can be embedded in a plane such that all edges are straight line segments

Definition 3.6.3. A graph is ployhedral if its vertices and edges may be identified with the vertices and edges of a convex polyhedron in the three dimensional space.

Theorem 3.6.4. A graph is polyhedral if and only if it is planar and 3 connected.

Theorem 3.6.5. Every polyhedron that has at last two faces with the same number of edges on the boundary.

Proof. The corresponding graph G is 3 connected. Hence $\delta(G) \geq 3$ and the number of faces adjacent to any chosen face f is equal to the number of edges in the boundary of the face f (if two faces have the edges u and vw with $r \neq w$ in common, then $G - \{r, w\}$ is disconnected contradicting 3 connectedness). Let $f_1, f_2, \ldots, f_m$ be the faces of the polyhedron and e_i be the number of edges on the boundary of the ith face. Let the faces be labelled so that $e_i \leq e_{i+1}$ for every i. If no two faces have the same number of edges in their boundaries, then $e_{i+1} - e_i \geq 1$ for every i. Hence $e_m - e_1 = \sum_{i=1}^{m-1}(e_{i+1} - e_i) \geq m - 1$ so that $e_m \geq e_1 + m - 1$. Since $e_1 \geq 3$, this implies that $e_m \geq m + 2$ so that the mth face is adjacent to at least $m+2$ faces. This gives a contradiction as there are only m faces. This proves the theorem. □

Theorem 3.6.6 (Euler Theorem)**.** If G is a connected plane graph having V, E, and F as the set of vertices, edges and faces respectively, then $|V| - |E| + |F| = 2$.

Proof. The proof is by induction on the number of edges of G. Let $|E| = 0$. Since G is connected, it is K_1 so that $|V| = 1$, $|F| = 1$ and hence $|V| - |E| + |F| = 2$. Now let G be a graph as in theorem and suppose that the theorem is true for all connected plane graphs with at most $|E| - 1$ edges.

If G is a tree, then $|E| = |V| - 1$ and $|F| = 1$ and hence $|V| - |E| + |F| = 2$. If G is not a tree, let x be an edge contained in some cycle of G. Then $G' = G - x$ is a connected plane graph such that $|V(G')| = |V|$, $|E(G')| = |E| - 1$ and $|F(G')| = |F| - 1$. Hence by induction hypothesis $|V(G')| - |E(G')| + |F(G')| = 2$ so that $|V| - (|E| - 1) + |F| - 1 = 2$. Hence $|V| - |E| + |F| = 2$.

□

Theorem 3.6.7. If G is a plane (p, q) graph with r faces and k components then $p - q + r = k + 1$.

Proof. Consider a plane embedding of G such that the exterior face of each component contains all other components. Now let the ith component be a (p_i, q_i) graph with r_i faces for each i. By the theorem $p_i - q_i + r_i = 2$. Hence

$$\sum p_i - \sum q_i + \sum r_i = 2k \tag{3.5}$$

But $\sum p_i = p, \sum q_i = q$ and $\sum r_i = r + (k - 1)$. Since the infinite face is counted k times in $\sum r_i$, hence equation (3.5) gives $p - q + r + k - 1 = 2k$ so that $p - q + r = k + 1$. $\square$

Corollary 3.6.1. If G is a (p, q) plane graph in which every face is an n cycle then $q = n(p-2)/(n-2)$.

Proof. Every face is an n-cycle. Hence each edge lies on the boundary of exactly two faces. Let $f_1, f_2, \ldots, f_r$ be the faces of G. Therefore

$$2q = \sum_{i=1}^{r}(\text{number of edges in the boundary of the face } \ f_i) = nr$$

This implies that $r = 2q/n$. By Eulers formula $p - q + r = 2$. That is

$$\begin{aligned} p - q + 2q/n &= 2 \\ q(2/n - 1) &= 2 - pq \qquad\qquad = n(p-2)/(n-2) \end{aligned}$$

$\square$

Corollary 3.6.2. In any connected plane (p, q) graph $(p \geq 3)$ with r faces $q \geq 3r/2$ and $q \leq 3p - 6$.

Proof.

Case 1 Let G be a tree. Then $r = 1, q = p - 1$ and $p \geq 3$. Hence $q \geq 3r/2$ and $q \leq 3p - 6$ since $p - 1 \leq 3p - 6$ (as $p \geq 3$).

Case 2 Let G have a cycle. let f_i $i = 1, 2, \ldots, r$ be the faces of G. Since each edge lies on the boundary of almost two faces,

$$2q \geq \sum_{i=1}^{r}(\text{number of edges in the boundary of face } \ f_i)$$

That is,

$$2q \geq 3r$$

That is

$$q \leq 3r/2 \tag{3.6}$$

By Euler's formula, $p - q + r = 2$. Substituting for r in equation (3.6), we get $q \geq 3/2(2 + q - p)$. After simplification we get, $q \leq 3p - 6$.

□

Definition 3.6.4. A graph is called maximal planar if no line can be added to it without losing planarity. In a maximal planar graph, each face is a triangle and such a graph is sometimes called a triangulated graph.

Corollary 3.6.3. If G is a maximal planar (p, q) graph then $q = 3p - 6$.

Corollary 3.6.4. If G is a plane connected (p, q) graph without triangles and $p \geq 3$, then $q \leq 2p - 4$.

Proof. If G is a tree, then $q = p - 1$. Hence we have $p - 1 = q \leq 2p - 4$. Now let G have a cycle. Since G has no triangles, the boundary of each face has at least four edges. Since each edge lies on at most two faces we have, $2q \geq \sum_{i=1}^{r}$(number of edges in the boundary of the ith face). That is,

$$2q \geq 4r. \tag{3.7}$$

By Euler's formula, we have $p - q + r = 2$. Substituting for r in equation (3.7), we get $2q \geq 4(2 + q - p)$. Hence $4p - 8 \geq 2q$ so that $q \leq 2p - 4$. □

Corollary 3.6.5. The graphs K_5 and $K_{3,3}$ are not planar.

Proof. Note that K_5 is a $(5, 10)$ graph. For any planar (p, q) graph, $q \leq 3p - 6$. But $q = 10$ and $p = 5$ do not satisfy this inequality. Hence K_5 is not planar. Also note that $K_{3,3}$ is a $(6, 9)$ bipartite graph and hence has no triangles. If such a graph is planar, then by Corollary refq12, $q \leq 2p - 4$. But $p = 6$ and $q = 9$ do not satisfy this inequality. Hence $K_{3,3}$ is not planar.

□

Corollary 3.6.6. Every planar graph G with $p \geq 3$ points has at least three points of degree less than 6.

By Corolary 3.6.2, $q \leq 3p - 6$. That is, $2q \leq 6p - 12$. That is, $\sum d_i \leq 6p - 12$ where d_i are the degrees of the vertices of G. Since G is connected, $d_i \leq 1$ for every i. If at most two d_i are less than 6, then $\sum d_i \geq 1 + 1 + 6 + \ldots + (p - 2) = 6p - 10$ which is a contradiction. Hence $d_i < 6$ for at least three values of i.

Theorem 3.6.8. Every planar graph G with at least 3 points is a subgraph of a triangulated graph with the same number of points.

Proof. Let G have p vertices. If $p \leq 4$, then G must be a subgraph of K_p which is a triangulated graph. Hence let $p \geq 5$.

We construct a triangulated graph G' which contains G as a subgraph as follows:

Consider a plane embedding of G. If R is a face of G and v_1 and v_2 are two vertices on the boundary of R without a connecting edge we connect v_1 and v_2 with an edge lying entirely in R. This yields a new plane graph. This yields a new plane graph. This operation is continued until every pair of vertices on the boundary of the same face are connected by an edge. The number of vertices remains the same under these operation. Hence the process terminates after some time yielding a plane triangulated graph G'. G is obviously a subgraph of G'. □

3.6.1 Characterization of Planar Graphs

Definition 3.6.5. Let $x = uv$ be an edge of a graph G. Line x is said to be subdivided when a new point w is adjoined to G and the line x is replaced by the lines uw and wv. This process is also called an elementary subdivision of the edge x. Two graphs are called homeomorphic if both can be obtained from the same graph by a sequence of subdivisions of the lines.

Example 3.6.2. Any two cycles are homeomorphic.

Theorem 3.6.9 (Kuratowski Theorem)**.** A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

Remark 3.6.1. The graphs K_5 and $K_{3,3}$ are called Kuratowski's graphs.

Definition 3.6.6. Let u and v be two adjacent points in a graph G. The graph obtained from G by the removal of u and v and the addition of a new point w adjacent to those points to which u or v was adjacent is called an elementary contraction of G. A graph G is contractible to a graph H if H can be obtained from G by a sequence of elementary contractions.

Example 3.6.3. The Petersen graph given in figure 3.5 is contractible to K_5 by contracting the lines $1a$, $2b$, $3c$, $4d$ and $5e$.

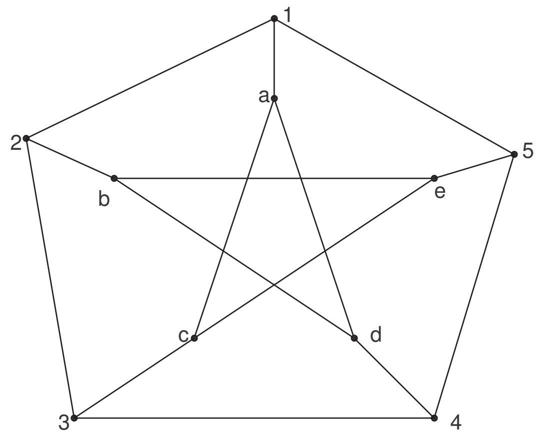

Figure 3.5: Petersen Graph

Theorem 3.6.10. A graph is planar if and only if it does no have a subgraph contractible to K_5 or $K_{3,3}$.

Since the Petersen graph is contractible to K_5, it is not planar according to the theorem 3.6.10.

Definition 3.6.7. Given a plane graph G, its geometrical dual G^* is constructed as follows: Place a vertex in each face of G(including the exterior face). For each edge x of G, draw an edge x^* joining the vertices representing the faces on both sides x such that x^* crosses only the edge x. The result is always a plane graph G^*(possibly with loops and multiple edges).

Module 4

Colourability

4.1 Chromatic Number and Chromatic Index

Definition 4.1.1. An assignment of colours to the vertices of a graph so that no two adjacent vertices get the same colour is called a *colouring* of the graph. For each colour, the set of all points which get that colour is independent and is called a colour class. A colouring of a graph G using at most n colours is called an n *colouring*. The *chromatic number* $\chi(G)$ of a graph $G$is the minimum number of colours needed to colour G. A graph G is called n-colourable if $\chi(G) \leq n$.

Example 4.1.1. The chromatic numbers of some well known graphs are given below:

Graph	K_p	$K_p - x$	$\overline{K_p}$	$K_{m,n}$	C_{2n}	C_{2n+1}
$\chi(G)$	p	$p-1$	1	2	2	3

Example 4.1.2. If T is a tree with at least two points, then $\chi(T) = 2$.

Example 4.1.3. Let W be a wheel. Then $\chi(W)$ is 3 or 4 according as it has an odd or even number of points.

Definition 4.1.2. Each n-colouring of G partitions $V(G)$ into independent sets called colour classes. Such a partitioning induced by a $\chi(G)$ colouring of G is called a chromatic partitioning. In other words, a partition of $V(G)$ into smallest possible number of independent sets is called a chromatic partitioning of G.

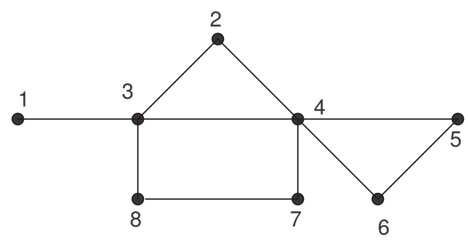

Figure 4.1: A graph with $\chi(G) = 3$

Example 4.1.4. Consider the graph shown in figure 4.1. Note that $chi(G) = 3$. $\{1, 4, 8\}$, $\{3, 6, 7\}$, $\{2, 5\}$ is a chromatic partitioning of this graph.

Theorem 4.1.1. Let G be any graph. Then the following statements are equivalent.

1. G is 2-colourable.
2. G is bipartite.
3. every cycle of G has even length.

Proof.

(1) $\Rightarrow$ (2) Assume that G is 2-colourable. Then $V(G)$ can be partitioned into two colour classes. These colour classes are independent sets and hence a partition of G. Hence G is bipartite.

(2) $\Rightarrow$ (1) Assume that G is bipartite. Then $V(G)$ can be partitioned into two sets V_1 and V_2 such that V_1 and V_2 are independent sets. A 2-colouring of G can be obtained by colouring all the points of V_1 white and all the points of V_2 blue. Hence G is 2-colourable.

(2) $\Leftrightarrow$ (3) (See Theorem 4.7, page)

□

Remark 4.1.1. G is bipartite does not imply that $\chi(G) = 2$. Consider the graph $\overline{K_2}$. Note that $\overline{K_2}$ is bipartite and $\chi(\overline{K_2}) = 1$.

Definition 4.1.3. A graph G is called *critical* if $\chi(H) < \chi(G)$ for every proper subgraph H of G. A k-chromatic graph that is critical is called k- critical. It is obvious that every k-chromatic graph has a k- critical subgraph.

Theorem 4.1.2. If G is k-critical, then $\delta(G) \geq k - 1$.

Proof. Since G is k-critical, for any vertex v of G, $\chi(G - v) = k - 1$. If $\deg(v) < k - 1$, then a $(k - 1)$ colouring of $G - v$ can be extended to a $k - 1$ colouring of $G$by assigning to v, a colour which is assigned to none of its neighbours in G. Hence $\deg(v) \geq k - 1$, so that $\delta(G) \geq k - 1$. □

Corollary 4.1.1. Every k-chromatic graph has at least k vertices of degree at least $k - 1$.

Proof. Let G be a k-chromatic graph and H be a k-critical subgraph of G. By theorem 4.1.2, $\delta(H) \geq k - 1$. Also since $\chi(H) = k$, H has at least k vertices. Hence H has at least k vertices of degree at least $k - 1$. Since H is a subgraph of G, the result follows. □

Corollary 4.1.2. For any graph G, $\chi \leq \Delta + 1$.

Proof. Let G have chromatic number χ. Let H be a χ- critical subgraph of G. By theorem 4.1.2, $\delta(H) \geq \chi - 1$. Hence $\chi \leq \delta(H) + 1$. Since $\delta(H) \leq \Delta(G)$, this implies that $\chi \leq \Delta(G) + 1$.

□

Theorem 4.1.3. For any graph G, $\chi(G) \leq 1 + \max\delta(G')$ where the summation is taken over all induced subgraphs G' of G.

Proof. The theorem is obvious for totally disconnected graphs. Now let G be an arbitrary n- chromatic graph, $n \geq 2$. Let H be any smallest induced subgraph of G such that $\chi(H) = n$. Hence $\chi(H - v) = n - 1$ for every point v of H. If $\deg_H v < n - 1$, then a $(n - 1)$ colouring of $H - v$ can be extended to a $n - 1$ colouring of H by assigning to v, a colour which is assigned to none of its neighbours in H. Hence $\deg_H v \geq n - 1$. Since v is an arbitrary vertex of H, this implies that $\delta(H) \geq n - 1 = \chi(G) - 1$.

Hence $\chi(G) \leq 1 + \delta(H) \leq 1 + \max\delta(H')$ where the maximum is taken over the set B of induced subgraphs G' of G. □

Definition 4.1.4. If$\chi(G) = n$ and every n-colouring of G induces the same partition on $V(G)$ then G is called *uniquely n-colourable* or *uniquely colourable.*

Example 4.1.5. K_3 and $K_4 - x$ are uniquely 3-colourable. K_n is uniquely n-colourable. $K_n - x$ is uniquely $(n-1)$ colourable. Any connected bipartitle graph is uniquely 2-colourable.

Theorem 4.1.4. If G is uniquely n-colourable, then $\delta(G) \geq n - 1$.

Proof. Let v be any point of v. In any n-colouring, v must be adjacent with at least one point of every colour different from that assigned to v. Otherwise, by reclouring v with a colour which none of its neighbours is having, a different n-colouring can be achieved. Hence degree of v is at least $(n - 1)$ so that $\delta(G) \geq n - 1$.

□

Theorem 4.1.5. Let G be a uniquely n-colourable graph. Then in any n-colouirng of G, the subgraph induced by the union of any two colour class is connected.

Proof. If possible, let C_1 and C_2 be two classes in a n-colouring of G such that the subgraph induced by $C_1 \cup C_2$ is disconnected. Let H be a component of the subgraph induced by $C_1 \cup C_2$. Obviously, no point of H is adjacent to a point in $V(G) - V(H)$ that is coloured C_1 or C_2. Hence interchanging the colours of the points in H and retaining the original colours for all other points, we get a different n-colouring for G. This gives a contradiction. □

Theorem 4.1.6. Every uniquely n-colourable graph is $(n - 1)$- connected.

Proof. Let$G$ be a uniquely n-colourable graph. Consider an n-colouring of G. If possible, let G be not $(n - 1)$ connected. Hence there exits a set S of at most $n - 2$ points such that $G - S$ is either trivial or disconnected. If $G - S$ is trivial, then G has at most $n - 1$ points so that G is not uniquely n-colourable. $G - S$ has at least two components. In the considered n-colouring, there are at least two colours say c_1 and $c_2$that are not assigned to any point of S.

If every point in a component of $G - S$ has colour different from c_1 and c_2, then by assigning colour c_1 to a point of this component, we get a different n-colouring of G. Otherwise, by interchanging the colours c_1 and c_2 in a component of $G - S$, a different n-colouring of G is obtained. In any case, G is not uniquely n-colourable, giving a contradiction. Hence G is $(n-1)$ connected. □

Corollary 4.1.3. In any n-colouring of a uniquely n-colourable graph G, the subgraph induced by the union of any k colour classes, $2 \leq k \leq n$, is $(k-1)$ connected.

Proof. If the subgraph H induced by the union of any k colour classes, $2 \leq k \leq n$, had different k-colourings then these k-colourings will induce different n-colourings for G giving a contradiction. Hence H is uniquely k-colourable. Hence by theorem 4.1.2 H is $(k-1)$ connected.

□

Definition 4.1.5. An assignment of colours to the edges of a graph G so that no two adjacent edges get the same colour is called an edge colouring or line coluring of G. An edge colouring of G using n colours is called a n-edge coluring (or n- line colouring. The edge chromatic number(also called line chromatic number or chromatic index) $\chi'(G)$ is the minimum number of colours needed to edge colour G. A graph G is called n-edge colourable if $\chi'(G) \leq n$.

Theorem 4.1.7. For any graph G, the edge chromatic number is either Δ or $\Delta + 1$.

Theorem 4.1.8. $\chi'(K_n) = n$ if n is odd $(n \neq 1)$ and $\chi'(K_n) = n - 1$ if n is even.

Proof. If $n = 2$, the result is obvious. Hence let $n > 2$. Let n be odd. Now the edges of K_n can be n-coloured as follows.

Place the vertices of K_n in the form of a regular n-gon. Colour the edges around the boundary using a different colour for each edge.

Let x be any one of the remaining edges. x divides the boundary into two segments, one say B_1 containing an odd number of edges and other containing an even number of edges. Colour x with the same colours as the edge that occurs in the middle of B_1. Note that these two edges are parallel. The result is a n-edge colouring of K_n since any two edges having the same colour are parallel and hence are not adjacent. Hence

$$\chi'(K_n) \leq n. \tag{4.1}$$

Since K_n has n points and n is odd, it can have at most $(n-1)/2$ mutually independent edges. Hence each colour class can have at most $(n-1)/2$ edges, so that the number of colour classes is at least $\binom{n}{2}\frac{1}{2}(n-1) = n$ so that

$$\chi'(K_n) \geq n \tag{4.2}$$

Equations (4.1) and (4.2) imply $\chi'(K_n) = n$.

Let $n(\geq 4)$ be even. Let K_n have vertices $v_1, v_2, \ldots, v_n$. Colour the edges of the subgraph K_{n-1} induced by the first $n-1$ points using the method described above. In this colouring, at each vertex, one colour(the colour assigned to the edge opposite to this vertex on the boundary) will be missing. Also, these missing colours are different. This edge colouring of K_{n-1} can be extended to an edge colouring of K_n by assigning the colour that is missing at v_i to edge v_iv_n for every $i, i < n$. Hence $\chi'(K_n) \leq n-1$. Also $\chi'(K_n) \geq \Delta(K_n) = n-1$. Hence $\chi'(K_n) = n-1$. □

Exercises

1. Give an example of a graph with $\Delta = \chi'$ and a graph with $\Delta < \chi'$.
2. Show that every outplanar graph is 3-colourable.
3. What is the smallest uniquely 3 colourable graph ?
4. What is the smallest uniquely 3 colourable graph which is not complete ?
5. Show that for any independent set S of points of a critical graph G, $\chi(G - S) = \chi(G) - 1$.
6. Show that the petersen graph has chromatic index 4.

4.2 The Five Colour Theorem

Heawood (1890) showed that one can always colour the vertices of a planar graph with at most five colours. This is known as the five colour theorem.

Theorem 4.2.1. Every planar graph is 5- colourable.

Proof. We will prove the theorem by induction on the number p of points. For any planar graph having $p \leq 5$ points, the result is obvious since the graph is p-colourable.

Now assume that all planar graphs with p points is 5- colourable for some $p \geq 5$. Let G be a planar graph with $p+1$ points. Then G has a vertex v of degree 5 or less. By induction hypothesis the plane graph $G - v$ is 5-colourable. Consider a 5-colouring of $G - v$ where c_i, $1 \leq i \leq 5$, are the colours used. If some colour, say c_j is not used in colouring vertices adjacent to v, then by assigning the colour c_j to v the 5 colouring of $G - v$ can be extended to 5-colouring of G.

Hence we have to consider only the case in which $\deg v = 5$ and all the five colours are used for colouring the vertices adjacent to v. Let v_1, v_2, v_3, v_4, v_5 be the vertices adjacent to v coloured c_1, c_2, c_3, c_4 and c_5 respectively.

Let G_{13}denote the subgraph of $G - v$ induced by those vertices coloured c_1 or c_3. If v_1 and v_3 belong to different components of G_{13}, then a 5 colouring of $G - v$ can be obtained by interchanging the coloures of vertices in the component of G_{13} containing v_1 (Since no point of this component is adjacent to a point with colour c_1 or c_3 outside this component, this interchange of colours results in a colouring of $G - v$. In this 5 colouring no vertex adjacent to v is coloured c_1, and hence by colouring v with c_1, a 5-coloring of G obtained.

If v_1 and v_3 are in the component of G_{13}, then in G there exits a $v_1 - v_3$ path of all of whose points are coloured c_1 or c_3. Hence there is no $v_2 - v_4$ path all whose points are colouredc_2, c_4.

Hence if G_{24} denotes the subgraph of $G - v$ induced by the points coloured c_2 or c_4, then v_2 and v_4 belong to different components of G_{24}. Hence if we interchange the colours of the points in the component of G_{24} containing v_2, a new colouring $G - v$ results and in this, no point adjacent to v is coloured c_2. Hence by assigning colour c_2 to v, we can get a 5-colouring of G. This completes the induction and the proof. □

4.3 Chromatic Polynomials

Birkhoff(1912) introduced chromatic polynomials as a possible means of attacking the four colour conjecture. This concept considers the number of ways

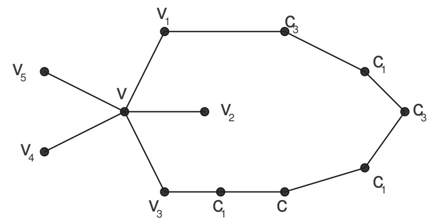

Figure 4.2: A graph explaining 5 colour theorem

of colouring a graph with a given number of colours.

Let G be a labeled graph. A colouring of G from λ colours is a colouring of G which uses λ or fewer colours. Two colourings of G from λ colours will be considered different if at least one of the labeled points is assigned different colours. Let $f(G, \lambda)$ denote the number of different colourings of G from λ colours. For example $f(K_1, \lambda) = \lambda$ and $f(\overline{K_2}, \lambda) = \lambda^2$.

Theorem 4.3.1. $f(K_n, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$

Proof. The first vertex in K_n can be coloured in λ different ways(as there are λ colours.) For each colouring of the first vertex, the second vertex can be coloured in $\lambda - 1$ ways (as there are $\lambda - 1$ colours remaining). For each colouring of the first two verties, the third can be coloured in $\lambda - 2$ ways and so on. Hence $f(K_n, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$. $\square$

Remark 4.3.1. $f(\overline{K_n}, \lambda) = \lambda^n$, since each of the n points of K_n may be coloured independently in λ ways.

Theorem 4.3.2. If G is a graph with k components $G_1, G_2, \ldots, G_k$, then $f(G, \lambda) = \prod_{i=1}^{n} f(G_i, \lambda)$

Proof. Number of ways of colouring G_i with λ colours is $f(G_i, \lambda)$. Since any choice of λ colouring for $G_1, G_2, \ldots, G_k$ can be combined to give a λ colouring for G, $f(G, \lambda) = \prod_{i=1}^{n} f(G_i, \lambda)$. $\square$

Definition 4.3.1. Let u and v be two nonadjacent points in a graph G. The graph obtained from G by removal of u and v and the addition of a new point w adjacent to those points to which u or v was adjacent is called an elementary homomorphism.

Theorem 4.3.3. If u and v are nonadjacent points in a graph G and hG denotes the elementary homomorphism of G which identifies u and v, then $f(G, \lambda) = f(G + uv, \lambda) + f(hG, \lambda)$ where $G + uv$ denotes the graph obtained from G by adding the line uv.

Proof.

$$
\begin{aligned}
f(G,\lambda) &= \text{number of colourings of } G \text{ from } \lambda \text{ colours} \\
&= (\text{number of colourings } G \text{ from } \lambda \text{ colours in which } u \text{ and } v \text{ get diffrent colours})+ \\
&\quad (\text{number of colurings of } G \text{ from } \lambda \text{ colurs in which } u \text{ and } v \text{ get the same colour}) \\
&= \text{number of colurings of } G+uv \text{ from } \lambda \text{ colurs}+ \\
&\quad \text{number of colurings of } hG \text{ from } \lambda \text{ colours} \\
&= f(G,\lambda) = f(G+uv,\lambda) + f(hG,\lambda)
\end{aligned}
$$

□

Corollary 4.3.1. Let G be a graph. Then

1. $f(G, \lambda)$ is a polynomial in λ.
2. $f(G, \lambda)$ has degree $|V(G)|$.
3. the constant term in $f(G, \lambda)$ is 0.

Proof. Theorem 4.3.3, states that $f(G, \lambda)$ can be written as the sum of $f(G_1, \lambda)$ and $f(G_2, \lambda)$ where G_1 has the same number of points as G with one more edge and G_2 has one point less than G. Doing this process repeatedly, $f(G, \lambda)$ can be written as $\sum f(G_i, \lambda)$ where each G_i is a complete graph and $\max|V(G_i)| = |V(G)|$.

Since $f(K_n, \lambda)$ is a polynomial of degree n, it follows that $f(G, \lambda)$ is a polynomial of degree $|V(G)|$. Since $f(K_n, \lambda)$ has constant term 0, the constant term in $\sum f(G_i, \lambda)$ is 0 so that (3) holds. □

Note 1. Because of the above corollary $f(G, \lambda)$ is called the chromatic polynomial of G.

The chromatic polynomial of a graph can be determined using theorem 4.3.1 as illustrated in the following example.

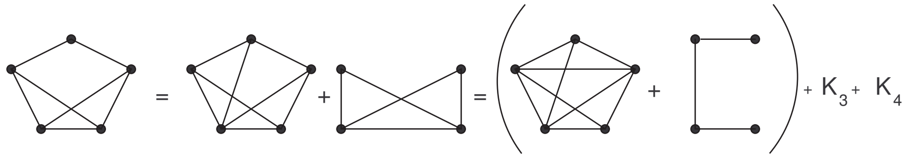

Figure 4.3: An example illusstrating the chromatic polynomial of a graph

Example 4.3.1. Find the chromatic polynomial of the graph G given in figure A diagram of the graph is used to denote the chromatic polynomial. The nonadjacent points considered at each step are indicated by u and v. Then

$$\begin{aligned} f(G,\lambda) &= [(K_5 + K_4) + (K_4 + K_3)] + (K_4 + K_3) \\ &= K_5 + 3K_4 + 2K_3 \\ &= f(K_5,\lambda) + 3f(K_4,\lambda) + 2f(K_3,\lambda) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 3\lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2) \\ &= \lambda^5 - 7\lambda^4 + 19\lambda^3 - 23\lambda^2 + 10\lambda. \end{aligned}$$

Theorem 4.3.4. If G is a tree with n points, $n \geq 2$, then $f(G,\lambda) = \lambda(\lambda-1)^{n-1}$.

Proof. We prove the result by induction on n. For $n = 2$, $G = K_2$ and hence $f(G,\lambda) = f(K_2,\lambda) = \lambda(\lambda-1)$ so that the theorem holds. Assume that the chromatic polynomial of any tree with $n-1$ points is $\lambda(\lambda-1)^{n-2}$. Let G be a tree with n points. Let v be an end point of G and let u be the unique point of G adjacent to v. By hypothesis, the tree $G-v$ has $\lambda(\lambda-1)^{n-2}$ for its chromatic. The point v can be assigned any colour different that assigned to u. Hence v may be coloured in $\lambda-1$ ways for each colouring of $G-v$. Thus

$$f(G,\lambda) = (\lambda-1)f(G-v,\lambda) = (\lambda-1)(\lambda-1)^{n-2} = \lambda(\lambda-1)^{n-1}$$

This complete the induction and the proof. □

The converse of the above theorem is given below:

Theorem 4.3.5. A graph G with n points and $f(G,\lambda) = \lambda(\lambda-1)^{n-1}$ is a tree.

Worked Examples

Problem 16. Prove that the coefficients of $f(G,\lambda)$ are alternate in sign.

We prove the result by induction on the number of lines q. When $q = 0$, $f(G,\lambda) = \lambda^p$ where p is the number of points of G. In this case the polynomial has just one non-zero coefficient and hence the result is trivially true.

Now assume that the result is true for all graphs with less than q lines. Let G be a (p,q) graph with $q > 0$. Let $e = uv$ be an edge of G. Let $G_1 = G - uv$. Clearly, u and v are nonadjacent in G_1. Hence

$$\begin{aligned} f(G_1,\lambda) &= f(G_1+uv,\lambda) + f(hG_1,\lambda) \\ &= f(G,\lambda) + f(hG_1,\lambda) \end{aligned}$$

Hence

$$f(G_1,\lambda) = f(G,\lambda) + f(hG_1,\lambda) \tag{4.3}$$

Now G_1 is a $(p,q-1)$ graph and hG_1 is a $(p-1,q_1)$ graph where $q_1 < q$. Hence by induction hypothesis

$$f(G_1,\lambda) = \lambda^p - \alpha_1\lambda^{p-1} + \alpha\lambda^{p-2} - \cdots + (-1)^{p-1}\alpha_{p-1}\lambda$$

and

$$f(hG_1,\lambda) = \lambda^{p-1} - \beta_1\lambda^{p-2} + \alpha\lambda^{p-2} - \cdots + (-1)^{p-1}\beta_{p-2}\lambda$$

where α_i and β_i are non negative integers. Hence by equation (4.3), we have

$$f(G,\lambda) = \lambda^p - (\alpha_1+1)\lambda^{p-1} + (\alpha_1+\beta_1)\lambda^{p-2} - \cdots + (-1)^{p-1}(\alpha_{p-1}+\beta_{p-2})\lambda$$

This is a polynomial in which the coefficients are alternate in sign.

Problem 17. Prove that if G is a (p,q) graph, the coefficient of λ^{p-1} in $f(G,\lambda)$ is $-q$.

We prove the result by induction on q. If $q = 0$ then $f(G,\lambda) = \lambda^p$. Hence the coefficient of λ^{p-1} is $-q$. Now assume that the result is true for all graphs with less than q edges. As in the previous problem,

$$f(G, \lambda) = f(G_1, \lambda) - f(hG_1, \lambda)$$

Since G_1 is a $(p, q-1)$ graph by induction hypothesis coefficient of λ^{p-1} in $f(G_1, \lambda) = -(q-1)$. Also, the coefficient of λ^{p-1} in $f(hG_1, \lambda) = 1$. Hence the coefficient of λ^{p-1} in $f(G, \lambda) = -(q-1) - 1 = -q$. This complete the induction and the proof.

Problem 18. Prove that $\lambda^4 - 3\lambda^3 + 3\lambda^2$ cannot be the chromatic polynomial of any graph.

Suppose there exits a graph G such that

$$f(G, \lambda) = \lambda^4 - 3\lambda^3 + 3\lambda^2$$

Therefore the number of points in G is 4. Also the number of lines in G is 3.

Case 1 Suppose G is connected. Since $q = 3 = p - 1$, G is a tree. Hence

$$f(G, \lambda) = \lambda(\lambda - 1)^3 = \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$$

which is a contradiction.

Case 2 Suppose G is not connected. Then $G = K_3 \cup K_1$. Therefore,

$$f(G, \lambda) = f(K_3, \lambda)f(K_1, \lambda) = \lambda(\lambda - 1)(\lambda - 2)\lambda = \lambda^4 - 3\lambda^3 + 3\lambda^2$$

which is again a contradiction.

4.4 Exercises

1. Find the chromatic polynomial of $K_4 - x$ where x is a line.

2. Show that $\lambda^4 - 3\lambda^3 + 5\lambda^2 - 1$ can not be the chromatic polynomial of a graph.

3. Prove that a graph G is connected iff the coefficients of λ in $f(G, \lambda)$ is not zero.

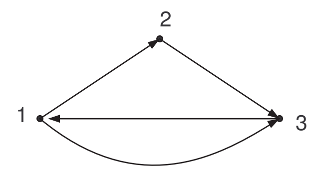

Figure 4.4: A digraph

4. Prove that if k is the least positive integer such that λ^k has non zero coefficients in $f(G, \lambda)$ than G is a graph with k components.

4.5 Directed graphs

Definition 4.5.1. A directed graph (digraph) D is a pair (V, A) where V is a finite non empty set and A is a subset of $V \times V - \{(x, x) : x \in V\}$. The elements of V and A are respectively called vertices(points) and arcs. If $(u, v) \in A$ then the arc (u, v) is said to have u as its initial vertex(tail) and v as its terminal vertex (head). Also the arc (u, v) is said to join u to v.

Just as graphs, digraphs can also be represented by means of diagrams. In these diagrams, vertices are denoted by points and arc (u, v) is represented by means of arrow from u to v. We shall often refer to the diagram of a digraph as the digraph itself.

Example 4.5.1. Let $V = \{1, 2, 3\}$ and $A = \{(1, 2), (2, 3), (1, 3), (3, 1)\}$. Then (V, A) is a digraph. The diagrammatic representation of this digraph is shown in figure 4.8.

Definition 4.5.2. The indegree $d^-(v)$ of a vertex v in a digraph D is the number of arcs having v as its terminal vertex. The outdegree $d^+(v)$ of v is the number of arcs having v as its initial vertex. The ordered pair $(d^+(v), d^-(v))$ is called the degree pair of v.

Consider the digraph shown in figure 4.8. The degree pairs of the points $1, 2, 3$ and 4 are $(2, 1), (1, 1), (1, 2)$ and $(0, 0)$ respectively.

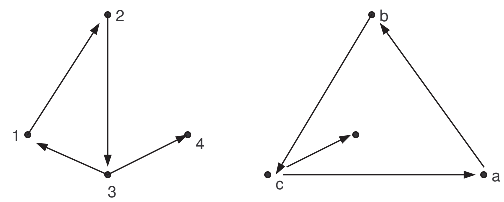

Figure 4.5: Two isomorphic digraphs

Theorem 4.5.1. In a digraph D, sum of all the indegrees of all the vertices is equal to the sum of their out degrees, each sum being equal to the number of arcs in D.

Proof. Let q denote the number of arcs in $D = (V, A)$. Let $B = \sum_{v \in V} d^+(v)$ and $C = \sum_{v \in V} d^-(v)$. An arc (u, w) contributes one to the out-degree of u and one to the in degree of w. Hence each arc contributes 1 to the sum B and 1 to the sum C. Hence $B = C = q$. □

Definition 4.5.3. A digraph $D' = (V', A')$ is called a subdigraph of $D = (V, A)$ if $V' \subseteq V$ and $A' \subseteq A$. The definition of induced subdigraph is analogous to that of induced subgraph. The underlying graph G of a digraph D is a graph having the same vertex set as D and two vertices u and v are adjacent in G whenever (u, w) or (w, u) is in A.

For example, consider the digraph (V, A) where $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2), (3, 4), (4, 3), (3, 2), (1, 4), (4, 1), (2, 4)\}$ has as its underlying graph. Similarly if we are given a graph G we can obtain a digraph from G by giving orientation to each edge of G. A digraph thus obtained from G is called an orientation of G.

Definition 4.5.4. Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are said to be isomorphic ($D_1 \simeq D_2$) if there exits a bijection $f : V_1 \to V_2$ such that $(u, w) \in A_1$ iff $(f(u), f(w)) \in A_2$. The function f is called an isomorphism from D_1 to D_2.

Example 4.5.2. Consider the digraphs shown in figure. These graphs are isomorphic. The isomorphism being $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = d$.

Theorem 4.5.2. If two digraphs are isomorphic then the corresponding points have the same degree pair.

Proof. Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be isomorphic under an isomorphism f. Let $v \in V_1$. Let

$$\begin{aligned} N(v) &= \{w : w \in V_1 \text{ and } (v, w) \in A_1\} \\ N(f(v)) &= \{w : w \in V_2 \text{ and } (f(v), f(w)) \in A_1\} \end{aligned}$$

Now

$$\begin{aligned} w \in N(v) &\Leftrightarrow (v, w) \in A_1 \\ &\Leftrightarrow (f(v), f(w)) \in A_2 \\ &f(w) \in N(f(v)). \end{aligned}$$

Hence $|N(v)| = |N(f(v))|$. This implies that v and $f(v)$ have the same out degree pair. Similarly, we can prove that v and $f(v)$ have the same in- degree pair. □

From theorems 4.5.1 and 4.5.2, it is obvious that two isomorphic digraphs have the same number of vertices and same number of arcs.

Definition 4.5.5. The converse digraph D' of a digraph D is obtained from D by reversing the direction of each arc.

Obviously D and D' have the same number of points and arcs. Moreover, the in-degrees of a point v in D is equal to its out-degree in D' and vice versa.

Definition 4.5.6. A digraph $D = (V, A)$ is called complete if for every pair of distinct points v and w in V, both (v, w) and (w, v) are in A.

Thus if a complete digraph has n vertices then it has $n(n-1)$ arcs.

Definition 4.5.7. A digraph is called functional if every point has out-degree one.

If a functional digraph has n vertices then the sum of the out-degrees of the points is n. Hence by theorem 4.5.1 the number of arcs in the digraph is n.

4.6 Path and Connectedness

Definition 4.6.1. A walk (directed walk) in a digraph is a finite alternating sequence $W = v_0x_1v_1 \dots x_nv_n$ of vertices and arcs in which $x_i = (v_{i-1}, v_i)$ for every arc x_i. W is called a walk from v_0 to v_n or a $v_0 - v_n$ walk. The vertices v_0 and v_n are called origin and terminus of W respectively and $v_1, v_2, \dots v_{n-1}$ are called its internal vertices. The length of a walk is the number of occurrence of arcs in it. A walk in which the origin and terminus coincide is called a closed walk. A path (directed path) is a walk in which all the vertices are distinct. A cycle (directed cycle or circuit) is a nontrivial closed walk whose origin and internal vertices are distinct.

If there is a path from u to v then v is said to be reachable from u.

Definition 4.6.2. A digraph is called strongly connected if every pair of points are mutually reachable. A digraph is called unilaterally connected or unilateral if for every pair of points, at least one is reachable from the other. A digraph is called weakly connected or weak if the underlying graph is connected. A digraph is called disconnected if the underlying graph is disconnected.

The trivial digraph consisting just one point is strong since it does not contain two distinct points. Obviously

$$\text{Strongly Connected } \Rightarrow \text{Unilaterally connected} \Rightarrow \text{weakly connected}$$

But the converse is not true.

Theorem 4.6.1. The edges of a connected graph $G = (V, E)$ can be oriented so that the resulting digraph is strongly connected iff every edge of G is contained in at least one cycle.

Proof. Suppose the edges of G can be oriented so that the resulting digraph becomes strongly connected.

If possible, let $e = vw$ be an edge of G not lying on any cycle. Now as soon as e is oriented, one of the vertices u and w becomes non reachable from the other. Hence an orientation of the required type is not possible, giving contradiction. Hence every edge of G lies on a cycle.

Conversely, let every edge of G lie on a cycle.

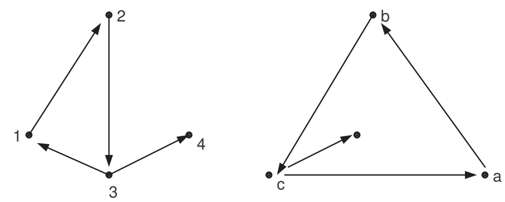

Figure 4.6: A digraph

Let $S = v_1, v_2, \ldots, v_n, v_1$ be a cycle in G. Orient the edges of S so that S becomes a directed cycle and hence becomes a strongly connected subdigraph. If $V = \{v_1, v_2, \ldots, v_n\}$ then we are through. Otherwise, let w be a vertex of G not in S such that w adjacent to a vertex v_i of S. Let $e = v_i w$. By hypothesis, e lies on some cycle C. We choose a direction of C and give the orientation determined by this direction to the edges of C which are not already oriented. The resulting enlarged oriented graph is also strongly connected as it can be got from S by a sequence of additionss of simple directed paths(For example, if $v \in S$ and u is a point on a simple directed $v_i - v_j$ path P added to S then the enlarged oriented graph the $u - v_j$ subpath of P followed by the $v_j - v$ subpath of S give a directed $u - v$ path. Also, the $v - v_i$ subpath of S followed by the $v_i - u$ subpath of P give a directed $v - u$ path. This type of argument can be repeated for each addition of simple directed paths) This process can be repeated till we get a strongly connected oriented spanning subgraph of G. The remaining edges now be oriented in any way. The resulting oriented graph is strongly connected. This completes the proof. $\square$

There are three diffrent kinds of components of a digraph.

Definition 4.6.3. Let $D = (V, A)$ be a digraph.

(a) Let W_1 be a maximal subset of V such that for every pair of points $u, v \in W_1$, u is reachable from v and v is reachable from u. Then the subdigraph of D induced by W_1 is called a strong component of D

(b) Let W_2 be a maximal subset of V such that for every pair of points $u, v \in W_2$, either u reachable from v or v is reachable from u. Then the subdigraph of D induced by W_2 is called a unilateral component of D.

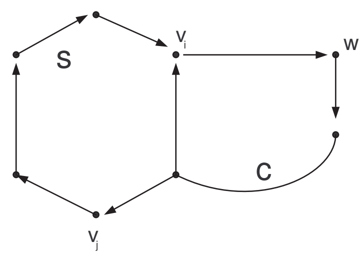

Figure 4.7: A digraph

(c) Let W_3 be a maximal subset of V such that for every pair of points $u, v \in W_3$, u and v are joined by a pat in the underlying graph of D. Then the subdigraph of D induced by W_3 is called a weak component of D.

Note 2. Let D be a digraph. Then each point of D is in exactly one strong component of D. An arc x lies in exactly one strong component if it lies on a cycle. There is no strong component containing an arc that does not lie on any cycle.

Example 4.6.1. Consider the digraph D shown in figure. The strong components are those subdigraphs induced by the sets of points $A = \{1, 2\}$,$B = \{3\}, C = \{4\},\ D = \{5\}, E = \{6, 7\}, F = \{8\}$ and $G = \{9, 10, 11, 12\}$. The unilateral components are those induced by the sets points

$$\{1, 2\}, \{3, 4, 6, 7, 8\}, \{4, 5\}, \{5, 6, 7\}, \{4, 6, 7, 9, 10, 11, 12\}, \{6, 7, 8, 9, 10, 11, 12\}$$

weak components are those induced by the sets of points

$$\{1, 2\}, \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Definition 4.6.4. The condensation D^* of a digraph D has the strong components $S_1, S_2, \ldots, S_n$ of D as its points with an arc from S_i to S_j whenever there is at least one arc from S_i to S_j in D.

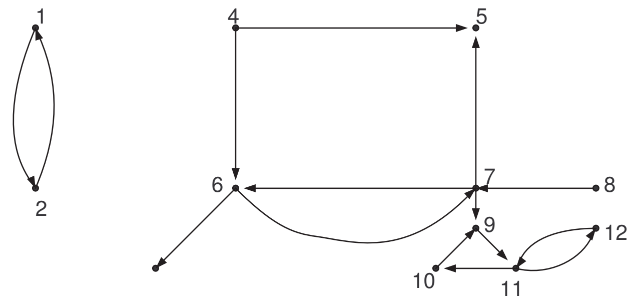

Figure 4.8: A digraph

Remark 4.6.1. If the condensation of a digraph has a cycle C then the strong components corresponding to points of C together form a strong component. This contradicts the maximality of strong components and hence the strong condensation has no cycles.

Definition 4.6.5. In a digraph D, a closed spanning walk in which each arc of D occurs exactly once is called an Eulerian trail(Euler Tour). A digraph is called Eulerian if it has an Eulerian trail.

Theorem 4.6.2. A weak digraph D is an Eulerian iff every point of D has equal in-degree and out-degree.

Proof. Let D be Eulerian and T be an Eulerian trail in D. Each occurrence(occurrence at origin and terminus of T together is to be considered as a single occurrence) of a given point in T contributes one to $d^-(v)$ and one to $d^+(v)$. Since each arc of D occurs exactly once in T, the contribution of each arc of D to $d^-(v)$ and $d^+(v)$ can be accounted in this way. Hence $d^-(v) = d^+(v)$ for every point v of D.

Conversely let $d^+(v) = d^-(v)$ for every point v of D. Since the trivial digraph is vacuously eulerian, let D have at least two points. Hence every point of D has positive in-degree and out-degree.

Hence D contains a cycle Z (Since if you reach a point for the first time, you can always move out). The removal of the lines of Z results in a spanning subdigraph D_1 in which again $d^-(v) = d^+(v)$ for every point v. If D_1 has no arcs, then Z is an eulerian trail in D. Otherwise, D_1 has a cycle Z_1. Continuing the above process when a digraph D_n with no arcs is obtained, we have a partition of the arcs of D into n cycles, $n \geq 2$. Among these n cycles, take two cycles Z_i and Z_j having a point in common. The walk beginning at v and consisting of the cycles Z_i and Z_j in succession is a closed trail containing the lines of these two cycles. Continuing this process, we construct a closed trail containing all the arcs of D. Hence D is eulerian. □

4.7 Exercise

1. Show that every eulerian digraph is strongly connected. Give an example to show that the converse is not true.

2. Show that a weak digraph D is eulerian iff the set of arcs of D can be partitioned into cycles.

3. Show that no strictly weak digraph contains a point whose removal results in a strong digraph.